Concomitants of Generalized Order Statistics for a Bivariate Weibull Distribution

Saman Hanif Shahbaz
Department of Statistics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia
saman.shahbaz17@gmail.com

Muhammad Qaiser Shahbaz
Department of Statistics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia
qshahbaz@gmail.com

Abstract
In this paper we have studied the distribution of \( r \)-th concomitant and joint distribution of \( r \)-th and \( s \)-th concomitant of generalized order statistics for a bivariate Weibull distribution. We have derived the expression for single and product moments. Numerical study has also been conducted to see the behavior of mean of concomitants for selected values of the parameters.

Keywords: Concomitants, Generalized Order Statistics, Bivariate Weibull distribution.

1. Introduction
A bivariate Weibull distribution is defined by Hanif Shahbaz and Ahmad (2009) and by Ahsanullah et al. (2010) as a compound distribution of two Weibull random variables. The density function of bivariate Weibull distribution defined by Hanif Shahbaz and Ahmad (2009) is

\[
f(x, y) = \beta \phi(x) \alpha_1 \alpha_2 x^{\alpha_1-1} y^{\alpha_2-1} \exp\left[-\left(\beta x^{\alpha_1} + \phi(x) y^{\alpha_2}\right)\right]; \\
\alpha_1 > 0, \alpha_2 > 0, \beta > 0, \phi(x) > 0, x > 0, y > 0,
\]

(1.1)

where \( \phi(x) \) is any positive real function of \( X \). Ahsanullah et al. (2010) have studied the distribution (1.1) for \( \phi(x) = x^{\alpha_1} \). The distribution in that case is given as

\[
f(x, y) = \beta \alpha_1 \alpha_2 x^{2\alpha_1-1} y^{\alpha_2-1} \exp\left[-x^{\alpha_1} \left(\beta + y^{\alpha_2}\right)\right]; \beta > 0, \alpha_1 > 0, \alpha_2 > 0, x > 0, y > 0. \quad (1.2)
\]

Distribution (1.1) can be studied for other choices of \( \phi(x) \). The distribution (1.2) has been study in context of order statistics and record values by Ahsanullah et al. (2010) and in context of order statistics by Hanif Shahbaz et al. (2011).

The generalized order statistics (gos) has been defined by Kamps (1995) as a unified model for ordered random variables. Kamps (1995) has argued that the quantities \( X_{r\,n,m,k} \) are called gos if their joint distribution is given as

\[
f_{1, \ldots, n, m, k}(x_1, x_2, \ldots, x_n) = k \left( \prod_{j=1}^{n-1} x_j \right) \left[ 1 - F(x_n) \right]^{k-1} f(x_n) \\
\times \left[ \prod_{j=1}^{n-1} \left( 1 - F(x_j)^m \right) \right] f(x_1). \quad (1.3)
\]
where \( n \) is sample size, \( m \) and \( k \) are parameters of the model and quantities \( \gamma_j \) are given as \( \gamma_j = k + (n-r)(m+1) \). The density function of a single \( gos \) is given by Kamps (1995) as

\[
f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x) \left[1 - F(x)\right]^{r-1} g_m^{-1} \left[F(x)\right], \tag{1.4}
\]

where \( C_{r-1} = \prod_{j=1}^{r} \gamma_j \); \( r = 1, 2, \ldots, n \), and

\[
g_m(x) = h_m(x) - h_m(0) = \begin{cases} \frac{1 - (1-x)^{m+1}}{(m+1)}; & m \neq -1 \\ -\ln(1-x); & m = -1. \end{cases}
\]

We also have

\[
h_m(x) = \begin{cases} -(1-x)^{m+1}/(m+1); & m \neq -1 \\ -\ln(1-x); & m = -1. \end{cases}
\]

Kamps (1995) has further shown that the joint density function of two \( gos \) \( X_{r,n,m,k} \) and \( X_{s,n,m,k} \) for \( r < s \) is given as

\[
f_{r,s,n,m,k}(x_1, x_2) = \frac{C_{s-r}}{(s-r-1)!} f(x_1) f(x_2) \left[1 - F(x_1)\right]^m g_m^{-1} \left[F(x_1)\right] \times \left[1 - F(x_2)\right]^{r-1} \left[h_m\left\{F(x_2)\right\} - h_m\left\{F(x_1)\right\}\right]^{r-s} ; -\infty < x_1 < x_2 < \infty. \tag{1.5}
\]

The density functions of \( gos \) given in (1.4) and (1.5) provide several models of ordered random variables as special case. Specifically for \( m = 0 \) and \( k = 1 \) the model reduces to Ordinary Order Statistics as given by David and Nagaraja (2003). Also for \( m = -1 \) we obtain \( s \)-th upper record values introduced by Chandler (1952). Other models like fractional order statistics given by Stigler (1977), sequential order statistics etc. can also be obtained for various values of the parameters involved. Other special cases of \( gos \) can be seen in Shahbaz et al. (2017).

Sometime it happen that a sample is available from a bivariate distribution, say \( F(x,y) \), and sample is arranged with respect to one of the variable, say \( X \). The other variable, \( Y \), is shuffled alongside the variable \( X \) and is called the concomitant of \( X \). When sample is arranged using order statistics then we have concomitants of order statistics and is discussed in David and Nagaraja (2003). Ahsanullah (1995) has discussed concomitants of record values. The concomitants of \( gos \) has been discussed by Ahsanullah and Nevzorov (2001) and by Shahbaz et al. (2017). The density function of \( r \)-th concomitant of \( gos \) is given as

\[
f_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f(y | x) f_{r,n,m,k}(x) \, dx, \tag{1.6}
\]

where \( f(y | x) \) is conditional distribution of \( Y \) given \( X = x \) and \( f_{r,n,m,k}(x) \) is defined in (1.4). The joint distribution of two concomitants is given as

\[
f_{[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1 | x_1) f(y_2 | x_2) f_{r,s,n,m,k}(x_1, x_2) \, dx_1 dx_2, \tag{1.7}
\]

where \( f_{r,s,n,m,k}(x_1, x_2) \) is given in (1.5).
Various authors have studied concomitants of $gos$. Concomitants of $gos$ for Gumbel Bivariate Exponential distribution has been studied by Ahsanullah and Beg (2006). Further Beg and Ahsanullah (2008) has studied concomitants of GOS for Gumbel bivariate family of distributions. Nayabuddin (2013) has studied concomitants of GOS for bivariate Lomax distribution. Hanif Shahbaz and Shahbaz (2016) have studied the concomitants of $gos$ for a bivariate exponential distribution.

In this paper we have obtained the distribution of the concomitants of upper record statistics for Bivariate Pseudo-Weibull distribution. Firstly, we have defined the Bivariate Pseudo-Weibull distribution in the following section.

2. Bivariate Pseudo-Weibull Distribution

The bivariate pseudo Weibull distribution has been defined by Hanif Shahbaz and Ahmad (2009) as compound distribution of Weibull random variables. The density function of bivariate Weibull distribution is given in (1.2). From the density function we can readily see that the marginal density function of $X$ is

$$f(x) = \beta \alpha_1 x^{\alpha_1-1} \exp\left(-\beta x^{\alpha_1}\right); \quad \beta > 0, \quad \alpha_1 > 0, \quad x > 0. \quad (2.1)$$

The conditional distribution of $Y$ given $X = x$ is

$$f(y|x) = \alpha_2 x^{\alpha_2} y^{\alpha_2-1} \exp\left(-x^{\alpha_2} y^{\alpha_2}\right); \quad \alpha_2 > 0, \quad \alpha_2 > 0, \quad x > 0, \quad y > 0. \quad (2.2)$$

The marginal and conditional distributions are useful in studying the distribution of concomitants of $gos$ for bivariate Weibull distribution.

In the following section the distribution of concomitant of record statistics has been derived for (1.2).

3. Distribution of $r$-th Concomitant and its Properties

The Bivariate Pseudo-Weibull distribution has been given in (1.1) and (1.2). In this section the distribution of $r$-th concomitants of $gos$ for Bivariate Pseudo-Weibull distribution, given in (1.2), has been obtained.

In order to obtain the distribution of concomitant of $gos$ we first need the distribution of $r$th $gos$ for the marginal distribution of $X$ given in (2.1). The distribution of $gos$ for $X$ can be obtained by using (1.4). For this we first see that

$$F(x) = 1 - \exp\left(-\beta x^{\alpha_1}\right); \quad \beta > 0, \quad \alpha_1 > 0, \quad x > 0.$$

Also

$$g_m[F(x)] = \frac{1}{m+1}\left[1 - \{1 - F(x)\}^{m+1}\right] = \frac{1}{m+1}\left[1 - \exp\left\{-(m+1)\beta x^{\alpha_1}\right\}\right].$$

So

$$g^{-1}_m[F(x)] = \frac{1}{(m+1)^{r-1}\left[1 - \exp\left\{-(m+1)\beta x^{\alpha_1}\right\}\right]^{r-1}}$$

$$= \frac{1}{(m+1)^{r-1}\sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i}\exp\left\{-(m+1)\beta ix^{\alpha_1}\right\}. \quad (3.1)$$
Now, using (2.1) and (3.1) in (1.4), the distribution of $r$th $gos$ for $X$ is

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x) \left[1 - F(x)\right]^{r-1} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \left(\frac{r-1}{i}\right) \exp\left(-\beta x^\alpha_i\right)$$

or

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)! \left(m+1\right)^{r-1}} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \left(\frac{r-1}{i}\right) \beta \alpha_i x^{\alpha_i-1} \exp\left(-\beta w_i x^{\alpha_i}\right),$$  \hspace{1cm} (3.2)

where $w_i = \{(m+1)i + \gamma_r\}$.

The conditional distribution of $Y$ given $X$ is given in (2.2). Now using (2.2) and (3.2) in (1.6), the distribution of $r$th concomitant of $gos$ for bivariate Weibull distribution is

$$f_{r,n,m,k}(y) = \int_{0}^{\infty} \alpha_2 x^{\alpha_2-1} \exp(-x^{\alpha_2}) \frac{C_{r-1}}{(r-1)! \left(m+1\right)^{r-1}} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \left(\frac{r-1}{i}\right) \beta \alpha_i x^{\alpha_i-1} \exp(-\beta w_i x^{\alpha_i}) dx$$

or

$$f_{r,n,m,k}(y) = \frac{C_{r-1}}{(r-1)! \left(m+1\right)^{r-1}} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \left(\frac{r-1}{i}\right) \beta \alpha_i y^{\alpha_i-1} \frac{1}{\left(y^{\alpha_i} + \beta w_i\right)^2}; y > 0.$$  \hspace{1cm} (3.3)

The distribution of concomitants for special cases when sample is available from a bivariate Weibull distribution can be obtained from (3.3) by using specific values of the parameters involved.

The $r$–th moment of the distribution given in (3.3) is obtained as:

$$\mu^p_{r,n,m,k} = \frac{C_{r-1}}{(r-1)! \left(m+1\right)^{r-1}} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \left(\frac{r-1}{i}\right) \int_{0}^{\infty} y^p \frac{\beta \alpha_i y^{\alpha_i-1}}{\left(y^{\alpha_i} + \beta w_i\right)^2} dy,$$

which after simplifications become

$$\mu^p_{r,n,m,k} = \frac{\beta p C_{r-1} \Gamma\left(p/\alpha_i\right) \Gamma\left(1-p/\alpha_i\right)}{\alpha_2 \left(r-1\right)! \left(m+1\right)^{r-1}} \sum_{i=0}^{r-1} \binom{r-1}{i} \left(\frac{r-1}{i}\right) (\beta w_i)^{p/\alpha_i-1},$$  \hspace{1cm} (3.4)

which exist for $p < \alpha_2$. We can see that the moment expression given in (3.5) reduces to expression for moments of concomitants of order statistics given by Shahbaz et al. (2009) for $m=0$ and $k=1$. The table of means for $m = 2$, $k = 2$ and for various values of $n$, $\beta$ and $\alpha_2$ is given below.
Table 1: Mean of $gos$ for Selected Values of Parameters

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha_2$</th>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>2.5</td>
<td>5</td>
<td>4.499</td>
<td>16.873</td>
<td>23.200</td>
<td>13.533</td>
<td>2.538</td>
<td>6</td>
<td>6.345</td>
<td>55.518</td>
<td>215.131</td>
<td>484.045</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>6.345</td>
<td>55.518</td>
<td>215.131</td>
<td>484.045</td>
<td>695.815</td>
<td>661.024</td>
<td>413.140</td>
<td>162.305</td>
<td>35.504</td>
<td>2.959</td>
</tr>
</tbody>
</table>

From above table we can see that the expected values shows an interesting trend. For even values of $n$, the mean increases for $1 \leq r \leq (n/2)$ and decreases for $(n/2) < r \leq n$. For odd values of $n$, the mean increases for $1 \leq r \leq (n+1)/2$ and decreases for $(n+1)/2 < r \leq n$. It can be further seen that for fixed value of $n$, the mean decreases with increase in $\beta$ and $\alpha_2$.

The mean and the variance of the concomitants of $gos$ for other values of parameters can also be tabulated.

The distribution function of $r$th concomitant of $gos$ for bivariate Weibull distribution is given as

$$F_{\gamma_{n,m,k}}(y) = \frac{C_{r-1}}{(r-1)(m+1)^{r-1}} \sum_{i=0}^{r-1}(-1)^i \left( \begin{array}{c} r-1 \\ i \end{array} \right) \int_0^y \frac{\beta \alpha_2 y^{\alpha_2-1}}{(y^{\alpha_2} + \beta w_i)^2} dy$$

$$= \frac{C_{r-1}}{(r-1)(m+1)^{r-1}} \sum_{i=0}^{r-1}(-1)^i \left( \begin{array}{c} r-1 \\ i \end{array} \right) \frac{y^{\alpha_2}}{w_i (y^{\alpha_2} + \beta w_i)}. \quad (3.5)$$

The hazard rate function for concomitant of $gos$ for bivariate Weibull distribution can be easily written by using (3.3) and (3.5) as

$$h_{\gamma_{n,m,k}}(y) = \frac{C_{r-1}}{(r-1)(m+1)^{r-1}} \sum_{i=0}^{r-1}(-1)^i \left( \begin{array}{c} r-1 \\ i \end{array} \right) \frac{\beta \alpha_2 y^{\alpha_2-1}}{(y^{\alpha_2} + \beta w_i)^2}; y > 0. \quad (3.6)$$
The hazard rate function can be computed for given values of the parameters involved. In the following section we have obtained the joint distribution of two concomitants of $gos$ for bivariate Weibull distribution.

4. Joint Distribution of the Concomitants and Moments

In this section we have derived the joint distribution of the concomitants of $gos$ for bivariate Weibull distribution given in (1.2). The joint distribution is obtained by using expression (1.7). In order to obtain the joint distribution we first obtain the joint distribution of two $gos$ by using (1.5) and is given as

$$f_{r,s,n,m,k}(x_1,x_2) = \frac{C_{r-1}}{(r-1)!(s-r-1)!} \beta \alpha_1 x_1^{\alpha_1-1} \exp\left(-\beta x_1^{\alpha_1}\right) \beta \alpha_2 x_2^{\alpha_2-1} \exp\left(-\beta x_2^{\alpha_2}\right) \exp\left(-\beta m x_i^{\alpha_i}\right)$$

$$\times \frac{1}{(m+1)^{r-1}} \sum_{i=0}^{r-1} \left(-1\right)^i \left(r-1\right) \exp\left(-\beta (m+1)i x_i^{\alpha_i}\right) \exp\left(-\beta x_i^{\alpha_i} \left(\gamma_i - 1\right)\right)$$

$$\times \frac{1}{(m+1)^{r-1}} \left[ \exp\left(-\beta (m+1)x_i^{\alpha_i}\right) - \exp\left(-\beta (m+1)x_j^{\alpha_j}\right) \right]^{r-1},$$

which after simplification becomes

$$f_{r,s,n,m,k}(x_1,x_2) = \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^2} \sum_{i=0}^{r-1} \sum_{j=0}^{r-1} \left(-1\right)^{i+j} \left(r-1\right) \left(s-r-1\right) \beta^2 \alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2-1} \exp\left(-\beta w_2 x_2^{\alpha_2}\right) \exp\left(-\beta w_1 x_1^{\alpha_1}\right); \quad 0 < x_1 < x_2 < \infty,$$

where $w_2 = \{(m+1)(s-r-j+i)\}; w_3 = \{(m+1)j + \gamma_i\}.$

Using the distribution (4.1) and (2.2) in (1.5), the joint distribution of two concomitants of $gos$ for bivariate Weibull distribution is

$$f_{r,s,n,m,k}(y_1,y_2) = \int_0^\infty \int_0^\infty \alpha_1 x_1^{\alpha_1-1} y_1^{\alpha_1-1} \exp\left(-x_1^{\alpha_1} y_1^{\alpha_1}\right) \alpha_2 x_2^{\alpha_2-1} y_2^{\alpha_2-1} \exp\left(-x_2^{\alpha_2} y_2^{\alpha_2}\right)$$

$$\times \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^2} \sum_{i=0}^{r-1} \sum_{j=0}^{r-1} \left(-1\right)^{i+j} \left(r-1\right) \left(s-r-1\right) \beta^2 \alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2-1} \exp\left(-\beta w_2 x_2^{\alpha_2}\right) \exp\left(-\beta w_1 x_1^{\alpha_1}\right) dx_1 dx_2$$

$$f_{r,s,n,m,k}(y_1,y_2) = \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^2} \sum_{i=0}^{r-1} \sum_{j=0}^{r-1} \left(-1\right)^{i+j} \left(r-1\right) \left(s-r-1\right) \beta^2 \alpha_1^2 x_1^{\alpha_1-1} x_2^{\alpha_2-1} \exp\left(-\beta w_2 x_2^{\alpha_2}\right) \exp\left(-\beta w_1 x_1^{\alpha_1}\right)$$

where $I(x_2) = \int_0^\infty x_1^{2\alpha_1-1} y_1^{\alpha_1-1} \exp\left(-x_1^{\alpha_1} \left(y_1^{\alpha_1} + \beta w_2\right)\right) dx_1$

$$I(x_2) = \int_0^\infty x_1^{2\alpha_1-1} y_1^{\alpha_1-1} \exp\left(-x_1^{\alpha_1} \left(y_1^{\alpha_1} + \beta w_2\right)\right) dx_1.$$

Using the transformation $x_1^{\alpha_1} \left(y_1^{\alpha_1} + \beta w_2\right) = t$ and simplifying we have

$$I(x_2) = \frac{y_1^{\alpha_1-1}}{\left(y_1^{\alpha_1} + \beta w_2\right)^2} \left[ 1 + x_1^{\alpha_1} \left(y_1^{\alpha_1} + \beta w_2\right) x_1 \right] \exp\left(-x_1^{\alpha_1} \left(y_1^{\alpha_1} + \beta w_2\right)\right).$$
Using this in (4.2) we have
\[
\begin{align*}
\mathcal{f}_{[r, s, m, a]}(y_1, y_2) = & \frac{C_{s-1}}{(r-1)(s-r-1)(m+1)^{s-2}} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\
& \times \beta^2 \alpha_1 \alpha_2 \frac{y_1^{\alpha_1-1} y_2^{\alpha_2-1}}{(y_2^{\alpha_2} + \beta w_3)^2} \int_0^\infty x_1^{\alpha_1-1} \{1 + x_1^{\alpha_1} (y_2^{\alpha_2} + \beta w_3) x_1\} \\
& \times \exp \left[ -\alpha_1 \left( y_1^{\alpha_1} + y_2^{\alpha_2} + \beta (w_2 + w_3) \right) \right] dx_1
\end{align*}
\]

Simplifying, the joint density function of two concomitants of gos for bivariate Weibull distribution is
\[
\begin{align*}
\mathcal{f}_{[r, s, m, a]}(y_1, y_2) = & \frac{\beta^2 \alpha_1 \alpha_2 C_{s-1}}{(r-1)(s-r-1)(m+1)^{s-2}} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \\
& \times \binom{s-r-1}{j} \left( y_1^{\alpha_1} + y_2^{\alpha_2} + \beta (w_2 + w_3) \right)^2 \{ y_1^{\alpha_1} + y_2^{\alpha_2} + \beta (w_2 + w_3) \}^{-3}
\end{align*}
\]

The product moments can be numerically computed from (4.3).

5. Conclusions and Recommendations

In this paper we have studied the distribution of concomitants of generalized order statistics when a sample is available from a bivariate Weibull distribution. The study has been conducted when \( \phi(x) = x^{\alpha_1} \). We have obtained the distribution of single concomitant and joint distribution of two concomitants. We have seen that the distribution of single concomitant of gos for bivariate Weibull distribution is weighted sum of Burr XII distributions. We have also seen that the mean of concomitants of gos increase with increase in the value of \( r \) until a specific point and then starts decreasing. This study can be extended by using some other choices of \( \phi(x) \).

Acknowledgement

The authors are thankful to anonymous reviewer for providing constructive comments which help improve the quality of the paper.

References


