Type II General Exponential Class of Distributions

G.G. Hamedani
Department of Mathematics, Statistics and Computer Science,
Marquette University, USA
zahid.uom1@gmail.com

Mahdi Rasekhi
Department of Statistics, Malayer University, Malayer, Iran
rasekhimahdi@gmail.com

Sayed M. Najibi
Department of Statistics, Persian Gulf University, Bushehr, Iran
mor.najibi@gmail.com

Haitham M. Yousof
Department of Statistics, Mathematics and Insurance,
Benha University, Benha, Egypt
haitham.yousof@fcom.bu.edu.eg

Morad Alizadeh
Department of Statistics, Persian Gulf University, Bushehr, Iran
moradalizadeh78@gmail.com

Abstract

In this paper, a new class of continuous distributions with two extra positive parameters is introduced and is called the Type II General Exponential (TIIGE) distribution. Some special models are presented. Asymptotics, explicit expressions for the ordinary and incomplete moments, moment residual life, reversed residual life, quantile and generating functions and stress-strengh reliability function are derived. Characterizations of this family are obtained based on truncated moments, hazard function, conditional expectation of certain functions of the random variable are obtained. The performance of the maximum likelihood estimators in terms of biases, mean squared errors and confidence interval length is examined by means of a simulation study. Two real data sets are used to illustrate the application of the proposed class.

Keywords: Maximum likelihood; Moment; Quantile Function, Order Statistics.

1.Introduction

Recently, several families of continuous univariate distributions have been constructed by extending common families of continuous models. These generalized distributions give more flexibility by adding one "or more" parameters to the baseline model. For examble, Gupta et al. (1998) proposed the exponentiated-G class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Many other classes can be cited such as the Marshall-Olkin-G family by Marshall and Olkin (1997), the T-X family by Alzaatreh et al. (2013), Kumaraswamy transmuted-G by Afify et al. (2016b), transmuted geometric-G by Afify et al. (2016a), beta transmuted-H by Afify et al. (2016c), Burr X-G by Yousof et al. (2016), The odd Lindley-G family of distributions by

Silva et al. (2016), exponentiated transmuted-G family by Merovci et al. (2016), odd-Burr generalized family by Alizadeh et al. (2016a) the complementary generalized transmuted poisson family by Alizadeh et al. (2016b), logistic-X by Tahir et al. (2016a), a new Weibull-G by Tahir et al. (2016b), the two-sided power-G class by Korkmaz and Genc (2016), the type I half-logistic family by Cordeiro et al. (2016a), the Zografos-Balakrishnan odd log-logistic family of distributions by Cordeiro et al. (2016b), the generalized odd log-logistic family by Cordeiro et al. (2016c), the beta dd log-logistic generalized family of distributions by Cordeiro et al. (2016d), the Kumaraswamy odd log-logistic family of distributions by Alizadeh et al. (2016) and a new generalized odd log-logistic family of distributions by Haghbin et al. (2016), among others.

Let $p(t)$ be the probability density function (pdf) of a random variable $T \in a, b]$ for $-\infty < a < b < \infty$ and let $W[G(x)]$ be a function of the cdf of a random variable X such that $W[G(x)]$ satisfies the following conditions:

$$\begin{cases} (i) & W[G(x)] \in a, b], \\ (ii) & W[G(x)] \text{differentiable and monotonically non} - \text{decreasing} \\ (iii) & W[G(x)] \to a \quad \text{as} \quad x \to -\infty \text{ and } W[G(x)] \to b \quad \text{as} \quad x \to \infty. \end{cases} \quad (1)$$

Recently, Alzaatreh et al. (2013) defined the T-X family of distributions by

$$F(x) = \int_a^{W[G(x)]} p(t)\, dt, \quad (2)$$

where $W[G(x)]$ satisfies conditions (1). Based on T-X idea, we define the cdf of Type **II** General Exponential (T**II**GE) class of distributions by

$$\begin{aligned} F_{\text{TIIGE}}(x) &= \lambda \int_0^{[G(x;\xi)^{-\alpha}-1]} \exp(-\lambda x) dt \\ &= 1 - \exp\left(\lambda\left\{1 - \left[\overline{G}(x;\xi)\right]^{-\alpha}\right\}\right),\ x \in \mathbb{R}, \end{aligned} \quad (3)$$

where $\xi = (\xi_k) = (\xi_1, \xi_2, \dots)$ is a parameter vector and $G(x;\xi) = G(x)$ is the baseline cdf and λ and α are positive parameters. For $\alpha = 1$, we obtain the odd exponential-G proposed by Bourguignon et.al. (2014). The corresponding pdf is

$$f_{\text{TIIGE}}(x) = \lambda\alpha g(x)\overline{G}(x)^{-(\alpha+1)} \exp\left(\lambda\left\{1 - \left[\overline{G}(x)\right]^{-\alpha}\right\}\right),\ x \in \mathbb{R}. \quad (4)$$

The reliability function (RF) $[R(X)]$, hazard rate function (HRF) $[h(X)]$, reversed-hazard rate function (RHR) $[r(x)]$ and cumulative hazard rate function (CHR) $[H(X)]$ of the T**II**GE family are given by

$$R(x) = \exp\left(\lambda\left\{1 - \left[\overline{G}(x)\right]^{-\alpha}\right\}\right),$$

$$h(x) = \lambda\alpha g(x)\overline{G}(x)^{-(\alpha+1)},$$

$$r(x) = \frac{\lambda\alpha g(x)\overline{G}(x)^{-(\alpha+1)} \exp\left(\lambda\left\{1 - \left[\overline{G}(x)\right]^{-\alpha}\right\}\right)}{1 - \exp\left(\lambda\left\{1 - \left[\overline{G}(x)\right]^{-\alpha}\right\}\right)},$$

and

$$H(x) = -\lambda\left\{1 - \left[\overline{G}(x)\right]^{-\alpha}\right\},$$

respectively. If $U \sim U(0,1)$ and $Q_G(.)$ denote the quantile function of G, then

$$X_U = G^{-1}\left\{1 - \left[1 - \frac{\ln(1-U)}{\lambda}\right]^{\frac{-1}{\alpha}}\right\},$$

has cdf (3). An interpretation of the T**II**GE family (3) can be given as follows. Let T be a random variable describing a stochastic system by the cdf $1 - \bar{G}(x)^{\alpha}$ (for $\alpha > 0$). If the random variable X represents the odds ratio, the risk that the system following the lifetime T will not be working at time x is given by $(1-\bar{G}(x)^{\alpha})/\bar{G}(x)^{\alpha}$. If we are interested in modeling the randomness of the odds ratio by the Exponential pdf $r(t) = \lambda e^{-\lambda t}$ (for $t > 0$), the cdf of X is given by

$$Pr(X \leq x) = R\left(\frac{(1-\bar{G}(x)^{\alpha})}{\bar{G}(x)^{\alpha}}\right),$$

which is exactly the cdf (3) of the new family.

The basic motivations for using the T**II**GE-G family in practice are: (i) to make the kurtosis more flexible compared to the baseline model; (ii) to produce a skewness for symmetrical distributions; (iii) to construct heavy-tailed distributions that are not longer-tailed for modeling real data; (iv) to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped; (v) to define special models with all types of the hrf; and (vi) to provide consistently better fits than other generated models under the same baseline distribution.

Now, Several structural properties of the extended distributions may be easily explored using mixture forms of exponentiated-G ("Exp-G") models. In the following, we obtain expansions for $F(x)$ and $f(x)$. So here we provide a useful representation for (3). The cdf of the T**II**GE family in (3) can be expressed as

$$F(x) = 1 - e^{\lambda - \lambda[\overline{G}(x)]^{-\alpha}} = 1 - \exp(\lambda) \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \overline{G}(x)^{\alpha j}. \tag{5}$$

Expanding $\overline{G}(x)^{-\alpha j}$ and after some algebra we get

$$F(x) = 1 - \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} \lambda^j}{j! \exp(-\lambda)} \binom{-\alpha\, j}{k} G(x)^k.$$

The above equation can be expressed as

$$F(x) = 1 - \sum_{k=0}^{\infty} a_k \mathbf{\Pi}_k(x), \tag{6}$$

where $a_k = \sum_{j=0}^{\infty} \frac{(-1)^{j+k} \lambda^j}{j! \exp(-\lambda)} \binom{-\alpha\, j}{k}$ and $\mathbf{\Pi}_\delta(x) = G(x)^\delta$ is the cdf of the Exp-G distribution with power parameter δ. The corresponding T**II**GE density function is obtained by differentiating (6)

$$f(x) = \sum_{k=0}^{\infty} v_{k+1} \pi_{k+1}(x), \tag{7}$$

where $v_{k+1} = -a_k$ and $\pi_\delta(x) = \delta g(x) G(x)^{\delta-1}$ is the pdf of the Exp-G distribution with power parameter δ. The properties of Exp-G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) and Mudholkar et al. (1995) for exponentiated Weibull, Gupta et al. (1998) for exponentiated Pareto, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for the exponentiated-type distributions, Nadarajah and Kotz (2006) for exponentiated Gumbel, Shirke and Kakade (2006) for exponentiated log-normal and Nadarajah and Gupta (2007) for exponentiated gamma distributions, among others.

The paper is unfolded as follows. In Section 2, we obtain some mathematical properties of the proposed model. In Section 3, we provide some useful characterizations of the new model. In Section 4, the model parameters are estimated by using maximum likelihood method and a simulation study is performed. Some special T**II**GE models are given in Section 5. Two applications are given in Section 6 to illustrate the flexibility of the proposed model. Finally, Section 7offers some concluding remarks.

2.Properties

2.1 Asymptotics

Let $a=\inf\{x|F(x)>0\}$. the asymptotics of cdf, pdf and hrf as $x\rightarrow a$ are given by

$$F(x)\sim\alpha\lambda G(x) \text{as} x\rightarrow a,$$
$$f(x)\sim\alpha\lambda g(x) \text{as} x\rightarrow a,$$
$$h(x)\sim\alpha\lambda g(x) \text{as} x\rightarrow a.$$

The asymptotics of cdf, pdf and hrf as $x\rightarrow\infty$ are given by

$$1-F(x)\sim\exp\left[-\lambda\overline{G}(x)^{-\alpha}\right] \text{as} x\rightarrow\infty,$$
$$f(x)\sim\alpha\lambda g(x)\overline{G}(x)^{-\alpha-1}\exp\left[-\lambda\overline{G}(x)^{-\alpha}\right] \text{as} x\rightarrow\infty,$$
$$h(x)\sim\alpha\lambda g(x)\overline{G}(x)^{-\alpha-1} \text{as} x\rightarrow\infty.$$

We can evaluate the effect of parameters on tails of distribution using these equations.

2.2 Moments and generating function

The rth moment of X, say μ'_r, follows from (7) as

$$\mu'_r=E(X^r)=\sum_{k=0}^{\infty}\upsilon_{k+1}\ E(Y_{k+1}^r). \quad (8)$$

Henceforth, Y_{k+1} denotes the Exp-G distribution with power parameter $(k+1)$. For $\gamma>0$, we have $E\left(Y_\gamma^r\right)=\alpha\ \int_{-\infty}^{\infty}x^r\ g(x;\xi)\ G(x;\xi)^{\gamma-1}\ dx$, which can be computed numerically in terms of the baseline quantile function (QF) $Q_G(u;\xi)=G^{-1}(u;\xi)$ as $E\left(Y_\gamma^n\right)=\gamma\ \int_0^1\ Q_G(u;\xi)^n\ u^{\gamma-1}du$. The nth central moment of X, say M_n, is given by

$$M_n=E(X-\mu'_1)^n=\sum_{r=0}^{n}\binom{n}{r}(-\mu'_1)^{n-r}\,E(X^r)$$
$$=\sum_{r=0}^{n}\sum_{k=0}^{\infty}(-1)^{n-r}\,\upsilon_{k+1}\ \binom{n}{r}\ (-\mu'_1)^{n-r}\,E(X^r).$$

The cumulants (κ_n) of X follow recursively from $\kappa_n=\mu'_n-\sum_{r=0}^{n-1}\binom{n-1}{r-1}\ \kappa_r\ \mu'_{n-r}$, where $\kappa_1=\mu'_1$, $\kappa_2=\mu'_2-\mu_1'^2, \kappa_3=\mu'_3-3\mu'_2\mu'_1+\mu_1'^3$, etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Here, we provide two formulas for the moment generating function (mgf) $M_X(t)=E(\mathrm{e}^{t\,X})$ of X. Clearly, the first one can be derived from equation (7) as $M_X(t)=\sum_{k=0}^{\infty}\upsilon_{k+1}\,M_{k+1}(t)$, where $M_{k+1}(t)$ is the mgf of Y_{k+1}. Hence, $M_X(t)$ can be determined from the Exp-G generating function. A second formula for $M_X(t)$ follows from (7) as $M_X(t)=\sum_{k=0}^{\infty}\upsilon_{k+1}\,\varsigma(t,k)$, where

$$\varsigma(t,k)=\int_0^1\exp[t\,Q_G(u)]\,u^k du$$

and $Q_G(u)$ is the qf corresponding to $G(x;\xi)$, i.e., $Q_G(u)=G^{-1}(u;\xi)$. For the T**II**GE-W model we obtain the following results

$$\mu'_r=\sum_{k,h=0}^{\infty}\upsilon_{k+1}\frac{(k+1)(-1)^i}{a^r(h+1)^{(r+b)/b}}\binom{k}{h}\ \Gamma\left(1+\frac{r}{b}\right),\forall r>-b,$$

and

$$M_X(t)=\sum_{k,r,h=0}^{\infty}\upsilon_{k+1}\frac{(k+1)(-1)^i t^r}{r!a^r(h+1)^{(r+b)/b}}\binom{k}{h}\ \Gamma\left(1+\frac{r}{b}\right),\forall r>-b.$$

Following similar algebraic developments of Nadarajah et al. (2013), the second formula for $M_X(t)$ can be obtained as follows: Using the series expansion

$$(1-z)^a = \sum_{h=0}^{\infty} \binom{a}{h} (-z)^h,$$

for $|z| < 1$, one can expand $(k+1)ba^{\beta}x^{b-1}\exp\left[-(ax)^{\,b}\right]\left\{1-\exp\left[-(ax)^{\,b}\right]\right\}^k$ as

$$f(x) = (k+1)\sum_{h=0}^{\infty} \frac{(-1)^h}{h+1}\binom{k}{h} f_{a(h+1)^{\frac{1}{b}}}(x), \tag{9}$$

where $f_{a(h+1)^{1/b}}(\cdot)$ denotes the pdf of a two-parameters Weibull model with a replaced by $(h+1)^{1/b}a$. So, whenever possible, (9) can be used to derive mgf of the T**II**GE-W model from those of a two-parameters Weibull distribution. Consider ${}_p\Psi_q(\cdot)$, the complex parameter Wright generalized hypergeometric function with p numerator and q denominator parameters (Kilbas *et al.*, 2006, Equation (1.9)) defined by the series

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j n)} \frac{z^n}{n!} \text{for} z \in \beta, \tag{10}$$

where α_j, β_k ∈Bbb , A_j, $B_k \neq 0$, $j = \overline{1,p}$, $k = \overline{1,q}$ and the series converges for $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$, compare with Mathai and Saxena (1978) and Srivastava et al. (1982). This function was originally introduced by Wright (1935). Let X be a random variable having the pdf (4), we can write the mgf of the T**II**GE-W model as

$$M_X(t) = \sum_{k,h=0}^{\infty} \upsilon_{k+1} \frac{(-1)^h (k+1)}{h+1} \binom{k}{h} {}_1\Psi_0\left[\begin{matrix}\left(1, \frac{1}{b}\right) \\ -\end{matrix}; t\,\{(h+1)^{\frac{1}{b}}a\}^{-1}\right]. \tag{11}$$

Hypergeometric functions are included as in-built functions in most popular algebraic mathematical software packages, so the special function in (10) and hence (11) can be easily evaluated by the software packages Maple, Matlab and Mathematica using known procedures.

2.3 Incomplete moments

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The sth incomplete moment, say $\varphi_s(t)$, of X can be expressed using (6) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{k=0}^{\infty} \upsilon_{k+1} \int_{-\infty}^{t} x^s \, \pi_{k+1}(x)dx. \tag{12}$$

The mean deviations about the mean $[\delta_1 = E(|X - \mu_1'|)]$ and about the median $[\delta_2 = E(|X - M|)]$ of X are given by $\delta_1 = 2\mu_1^{'} F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (3) and $\varphi_1(t)$ is the first incomplete moment given by (12) with $s = 1$. Now, we provide two ways to determine δ_1 and δ_2. First, a general equation for $\varphi_1(t)$ can be derived from $\varphi_s(t)$ as $\varphi_1(t) = \sum_{k=0}^{\infty} \upsilon_{k+1}\, \delta_{k+1}(x)$, where $\delta_{k+1}(x) = \int_{-\infty}^{t} x\, \pi_{k+1}(x)dx$ is the first incomplete moment of the exp-G distribution. A second general formula for $\varphi_1(t)$ is given by $\varphi_1(t) = \sum_{k=0}^{\infty} \upsilon_{k+1}\, \eta_k(t)$, where $\eta_k(t) = (k+1)\int_0^{G(t)} Q_G(u)\, u^k du$ can be computed numerically. These equations for $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(q)/(\pi\mu_1')$ and $L(\pi) = \varphi_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π. For the T**II**GE-W model we get

$$\varphi_s(t)=\sum_{k,h=0}^{\infty} \upsilon_{k+1}\frac{(k+1)(-1)^h}{a^s(h+1)^{(s+b)/b}}\binom{k}{h}\,\gamma\left(1+\frac{s}{b},\left(\frac{a}{t}\right)^b\right),\forall s>-b,$$

where $\gamma(.,.)$ is the lower incomplete gamma function.

2.4 moment residual life and reversed residual life

The nth moment of the residual life, say $z_n(t)=E[(X-t)^n|X>t],\ n=1,2,\ldots,$ uniquely determines $F(x)$. The nth moment of the residual life of X is given by $z_n(t)=\frac{1}{1-F(t)}\int_t^{\infty}(x-t)^n dF(x)$.

Therefore

$$z_n(t)=\frac{1}{1-F(t)}\sum_{k=0}^{\infty}\upsilon_{k+1}^{*}\int_t^{\infty}x^r\,\pi_{k+1}(x)dx,$$

where

$$\upsilon_{k+1}^{*}=\upsilon_{k+1}\sum_{r=0}^{n}\binom{n}{r}\,(-t)^{n-r}.$$

Another interesting function is the *mean residual life* (MRL) function or the life expectation at age t defined by $z_1(t)=E[(X-t)|X>t]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n=1$ in the last equation.

The nth moment of the reversed residual life, say $Z_n(t)=E[(t-X)^n|X\le t]$, for $t>0$ and $n=1,2,\ldots,$ uniquely determines $F(x)$. We obtain $Z_n(t)=\frac{1}{F(t)}\int_0^t(t-x)^n dF(x)$. Then, the nth moment of the reversed residual life of X becomes

$$Z_n(t)=\frac{1}{F(t)}\sum_{k=0}^{\infty}\upsilon_{k+1}^{**}\int_0^{t}x^r\,\pi_{k+1}(x)dx,$$

where

$$\upsilon_{k+1}^{**}=\upsilon_{k+1}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\,t^{n-r}.$$

The *mean inactivity time* (MIT), also called the mean reversed residual life function, is given by $Z_1(t)=E[(t-X)|X\le t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of the T**II**GE-G family can be obtained easily by setting $n=1$ in the above equation. For the T**II**GE-W model we get

$$z_n(t)=\frac{1}{1-F(t)}\sum_{k,h=0}^{\infty}\frac{(k+1)(-1)^h\upsilon_{k+1}^{*}}{a^n(h+1)^{(n+b)/b}}\binom{k}{h}\,\gamma\left(1+\frac{n}{b},\left(\frac{a}{t}\right)^b\right),\forall n>-b,$$

and

$$Z_n(t)=\frac{1}{F(t)}\sum_{k,h=0}^{\infty}\frac{(k+1)(-1)^h\upsilon_{k+1}^{**}}{a^n(h+1)^{\frac{(n+b)}{b}}}\binom{k}{h}\,\gamma\left(1+\frac{n}{b},\left(\frac{a}{t}\right)^b\right),\forall n>-b.$$

2.5 Order statistics

Suppose $X_1,\ldots,X_n$ is a random sample from any T**II**GE-G distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\,F(x)^{j+i-1}.$$

Following similar algebraic developments of Nadarajah et al. (2015), we can write the density function of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} b_{r,k}\, \pi_{r+k+1}(x), \tag{13}$$

where

$$b_{r,k} = \frac{n!\ (r+1)\ (i-1)!\ \upsilon_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, f_{j+i-1,k}}{(n-i-j)!\ j!},$$

υ_{r+1} is given in Section 1 and the quantities $f_{j+i-1,k}$ can be determined with $f_{j+i-1,0} = \upsilon_0^{j+i-1}$ and recursively for $k \geq 1$

$$f_{j+i-1,k} = (k\, \upsilon_0)^{-1} \sum_{m=1}^{k} [m\,(j+i)-k]\, \upsilon_m\, f_{j+i-1,k-m}.$$

Equation (13) is the main result of this section. It reveals that the pdf of the T**II**GE-G order statistics is a linear combination of exp-G density functions. So, several mathematical quantities of the T**II**GE-G order statistics such as ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the Exp-G distribution. For example, for the T**II**GE-W model we get

$$E\left(X_{i:n}^q\right) = \sum_{k,r,h=0}^{\infty} \frac{(r+k+1)(-1)^i b_{r,k}}{a^q (h+1)^{(q+b)/b}} \binom{r+k}{h}\ \Gamma\left(1+\frac{q}{b}\right), \forall q > -b.$$

2.6 Stress-strength model

The stress-strength model is the most widely approach used for reliability estimation. This model is used in many applications of physics and engineering such as strength failure and system collapse. In stress-strength modeling, $\mathbf{R} = \Pr(X_2 < X_1)$ is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1. The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. **R** can be considered as a measure of system performance and naturally raised in electrical and electronic systems. Other interpretations can be that, the reliability, say **R**, of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1 and X_2 be two independent random variables with T**II**GE$(\lambda_1, \alpha_1, \xi)$ and T**II**GE$(\lambda_2, \alpha_2, \xi)$ distributions. Then, the reliability is defined by

$$\mathbf{R} = \int_0^{\infty} f_1(x; \lambda_1, \theta_1, \alpha, \beta) F_2(x; \lambda_2, \theta_2, \alpha, \beta) dx = \sum_{k=0}^{\infty} \left(\Upsilon_{k+1} + \sum_{m=0}^{\infty} \phi_{k+m+2} \right),$$

where

$$\Upsilon_{k+1} = \sum_{j=0}^{\infty} \frac{(-1)^{j+k} \lambda_1^j}{j!\, e^{-\lambda_1}} \binom{-\alpha_1 j}{k},$$

and

$$\phi_{k+m+2} = \sum_{j,w=0}^{\infty} \frac{(-1)^{j+k+w+m+1} \lambda_1^j \lambda_2^w (k+1)}{j!\, w!\, e^{-\lambda_1-\lambda_2} (k+m+2)} \binom{-\alpha_1 j}{k} \binom{-\alpha_2\, w}{m}.$$

3. Characterizations

The problem of characterizing a distribution is an important problem, in its own right, which can help the investigator to see if their model is the correct one. This section deals with various characterizations of T**II**GE distribution. These characterizations are

presented in three directions: (i) based on the ratio of two truncated moments; (ii) in terms of the hazard function and (iii) based on the conditional expectation of certain functions of the random variable. It should be noted that characterization (i) can be employed also when the cdf does not have a closed form.
We present our characterizations $(i)-(iii)$ in three subsections.

3.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of TIIGE distribution based on the ratio of two truncated moments. Our first characterization employs a theorem of Glänzel (1987), see Theorem 1 of Appendix A .The result, however, holds also when the interval H is not closed since the condition of Theorem 1 is on the interior of H.

Proposition 3.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)\equiv 1$ and $q_2(x)=\exp\left\{\lambda\left(1-\left(\overline{G}(x;\xi)\right)^{-\alpha}\right)\right\}$ for $x\in\mathbb{R}$. The random variable X belongs to the family (4) if and only if the function η defined in Theorem1 has the form

$$\eta(x)=\frac{1}{2}\exp\left\{\lambda\left(1-\left(\overline{G}(x;\xi)\right)^{-\alpha}\right)\right\},\quad x\in\mathbb{R}.$$

Proof. Let X be a random variable with pdf (4), then

$$\left(1-F(x)\right)E[q_1(X)|X\geq x]=\exp\left\{\lambda\left(1-\left(\overline{G}(x)\right)^{-\alpha}\right)\right\},\quad x\in\mathbb{R},$$

and

$$\left(1-F(x)\right)E[q_2(X)|X\geq x]=\frac{1}{2}\exp\left\{2\lambda\left(1-\left(\overline{G}(x)\right)^{-\alpha}\right)\right\},\quad x\in\mathbb{R},$$

and finally

$$\eta(x)q_1(x)-q_2(x)=-\frac{1}{2}\exp\left\{\lambda\left(1-\left(\overline{G}(x)\right)^{-\alpha}\right)\right\}<0 for\ x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\alpha\lambda g(x)\left(\overline{G}(x)\right)^{-(\alpha+1)},\ x\in\mathbb{R},$$

and hence

$$s(x)=\lambda\left(\overline{G}(x)\right)^{-\alpha},\quad x\in\mathbb{R}.$$

Now, according to Theorem 1, X has density (4).

Corollary 3.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.1. Then, X has pdf (4) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)}{\eta(x)-g(x)}=\alpha\lambda g(x)\left(\overline{G}(x)\right)^{-(\alpha+1)},\quad x\in\mathbb{R}.$$

The general solution of the differential equation in Corollary 3.1 is

$$\eta(x)=\exp\left\{-\lambda\left(1-\left(\overline{G}(x)\right)^{-\alpha}\right)\right\}\begin{bmatrix}-\int\alpha\lambda g(x)\left(\overline{G}(x)\right)^{-(\alpha+1)}\times\\ \exp\left\{\lambda\left(1-\left(\overline{G}(x)\right)^{-\alpha}\right)\right\}q_2(x)dx+D\end{bmatrix},$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 3.1 with $D=0$.

3.2 Characterization in terms of the hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establishes a non-trivial characterization of T**II**GE in terms of the hazard function which is not of the above trivial form.

Proposition 3. 2. Let $X: \Omega \to \mathbb{R}$ be a continuous random variable. Then, X has pdf (4) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \frac{g'(x)}{g(x)} h_F(x) = \alpha(\alpha+1)\lambda(g(x))^2 \left(\overline{G}(x)\right)^{-(\alpha+2)},$$

with the boundary condition $h_F(0) = \alpha\lambda k(0)$.

Proof. If X has pdf (4), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{\left(g(x)\right)^{-1} h_F(x)\right\} = \alpha(\alpha+1)\lambda g(x)\left(\overline{G}(x)\right)^{-(\alpha+2)}$$

$$= \alpha\lambda \frac{d}{dx}\left\{\left(\overline{G}(x)\right)^{-(\alpha+1)}\right\},$$

or

$$h_F(x) = \alpha\lambda g(x)\left(\overline{G}(x)\right)^{-(\alpha+1)} \quad x > 0,$$

which is the hazard function of (4).

3.3 Characterization based on the conditional expectation of certain functions of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it as a proposition here, which can be used to characterize T**II**GE distribution.

Proposition 3.3. Let $X: \Omega \to (d, e)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (d, e) with $\lim_{x \to d^+} \psi(x) = 1$. Then for $\delta \neq 1$,

$$E[\psi(X) | X \geq x] = \delta\psi(x), \quad x \in (d, e),$$

if and only if

$$\psi(x) = \left(1 - F(x)\right)^{\frac{1}{\delta} - 1}, \quad x \in (d, e),$$

Remarks 3.1. For $(a, b) = \mathbb{R}$, $\psi(x) = \exp\left\{1 - \left(\overline{G}(x)\right)^{-\alpha}\right\}$ and $\delta = \frac{\lambda}{\lambda+1}$, Proposition 3.3 provides a characterization of T**II**GE.

4. Estimation

4.1 Maximum likelihood method

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used for constructing confidence intervals and also for test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. Here, we consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood. Let $x_1, \dots, x_n$ be a random sample from the TIIGE-G model with a $(q+2) \times 1$ parameter vector $\mathbf{\Phi}$ $=(\lambda, \alpha, \xi)^{ú}$, where ξ is a $q \times 1$ baseline parameter vector. The log-likelihood function for $\mathbf{\Phi}$ is given by

$$\ell_n(\mathbf{\Phi}) = n\log(\lambda) + n\log(\alpha) + \sum_{i=1}^{n} \log[g(x_i, \xi)] + \sum_{i=1}^{n} \log[\bar{G}(x_i, \xi)] + \lambda \sum_{i=1}^{n} \left\{1 - \left[\overline{G}(x;\xi)\right]^{-\alpha}\right\},$$

The log-likelihood function can be maximized by solving the following nonlinear normal equations

$$U(\lambda) = \frac{n}{\lambda} + \sum_{i=1}^{n} \left\{1 - \left[\overline{G}(x;\xi)\right]^{-\alpha}\right\}, U(\alpha) = \frac{n}{\alpha}] + \lambda \sum_{i=1}^{n} \left[\overline{G}(x;\xi)\right]^{-\alpha} \log\left[\overline{G}(x;\xi)\right],$$

and (for $r = 1, \dots, q$)

$$U(\xi_r) = \sum_{i=1}^{n} \frac{g^{(\xi_r)}(x_i, \xi)}{g(x_i, \xi)} - \sum_{i=1}^{n} \frac{G^{(\xi_r)}(x_i, \xi)}{\bar{G}(x_i, \xi)} - \lambda\alpha \sum_{i=1}^{n} G^{(\xi_r)}(x_i, \xi)\left[\overline{G}(x;\xi)\right]^{-\alpha-1}$$

Where

$$g^{(\xi_r)}(x_i, \xi) = \frac{\partial[g(x_i, \xi)]}{\partial \xi_r} \text{ and } G^{(\xi_r)}(x_i, \xi) = \frac{\partial[G(x_i, \xi)]}{\partial \xi_r}.$$

Setting the nonlinear system of equations $U(\lambda) = U(\alpha) = U(\xi_r) = 0$ (for $r = 1 = \cdots, q$) and solving them simultaneously yields the MLEs $\widehat{\mathbf{\Phi}} = (\hat{\lambda}, \hat{\alpha}, \hat{\xi}^{ú})^{ú}$. To solve these equations, it is more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell(\mathbf{\Phi})$. For interval estimation of the parameters, we can evaluate numerically the elements of the $(q+2) \times (q+2)$ observed information matrix $J(\mathbf{\Phi}) = \{-\frac{\partial^2}{\partial \Phi_r \, \Phi_s}[\ell_n(\mathbf{\Phi})]\}$. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\mathbf{\Phi}}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\mathbf{\Phi}})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\mathbf{\Phi}})$ is the total observed information matrix evaluated at $\widehat{\mathbf{\Phi}}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some sub-models of the TIIGE-G model. Hypothesis tests of the type $H_0: \omega = \omega_0$ versus $H_1: \omega \neq$

ω_0, where ω is a vector formed with some components of $\mathbf{\Phi}$ and ω_0 is a specified vector, can be performed using LR statistics. For example, the test of

$$H_0: \mathbf{\Phi} = 1 \text{ versus } H_1: H_0 \text{ is not true}$$

is equivalent to comparing the T**II**GE and G distributions and the LR statistic is given by $w = 2\{\ell(\hat{\lambda}, \hat{\alpha}, \hat{\xi}) - \ell(1,1,\tilde{\xi})\}$,where $\hat{\lambda}, \hat{\alpha}$ and $\hat{\xi}$ are the MLEs under H and $\tilde{\xi}$ is the estimate under H_0.

4.2 Simulation study

In this section, we survey the performance of the MLEs of the Type **II** General Exponential Lomax (T**II**GELO) distribution with respect to sample size n. This performance is done based on the following simulation study:

1.Generate 1000 samples of size n from T**II**GELO distribution. The inversion method was used to generate samples.

2.Compute the MLEs for 1000 thousand samples, say $(\hat{\alpha}, \hat{\lambda}, \hat{a}, \hat{b})$ for $i = 1,2, \ldots, 1000$ based on non-linear equations.

3.Compute the standard errors of the MLEs.

4.Compute the biases, mean squared errors and coverage lengths.

Then these steps are repeated for $n = 20,30,50,100,150,200,250,300$ T**II**GE-LO(15,2,20,1) and T**II**GE-LO(2,2,4,6), and computing $Bias(n), MSE(n), CI(n)$ for $\varepsilon = (\alpha, \lambda, a, b)$. Figure1 shows the variation of four parameter biases with respect to n. The biases for each parameter decrease to zero as n goes to infinity. Figure 2 shows how the four mean squared errors vary with respect to n. The mean squared errors for each parameter decrease to zero as $n \to \infty$. Table 1 shows the simulation study results for T**II**GE-LO(15,2,20,1) and T**II**GE-LO(2,2,4,6). In summary, the biases and MSEs for each parameter decreased to zero and appeared reasonably small at $n = 300$. Clearly, the rate of convergence of MSE for α and a is less than λ and b.

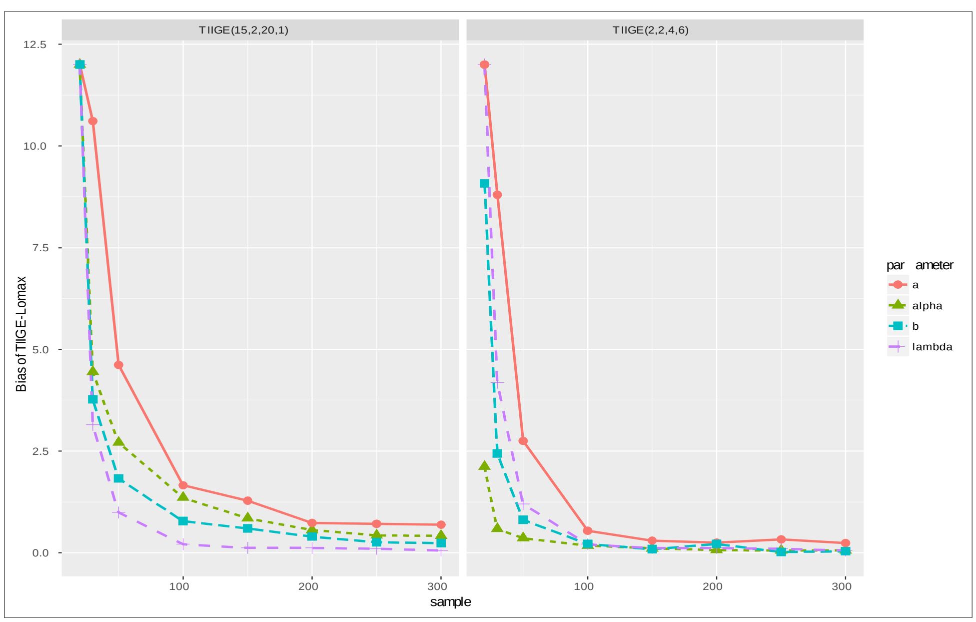


Figure 1: The Bias of different parameters of T**II**GE-LO(15,2,20,1) and T**II**GE-LO(2,2,4,6) versus n =20, 30, 50, 100, 150, 200, 250 and 300.

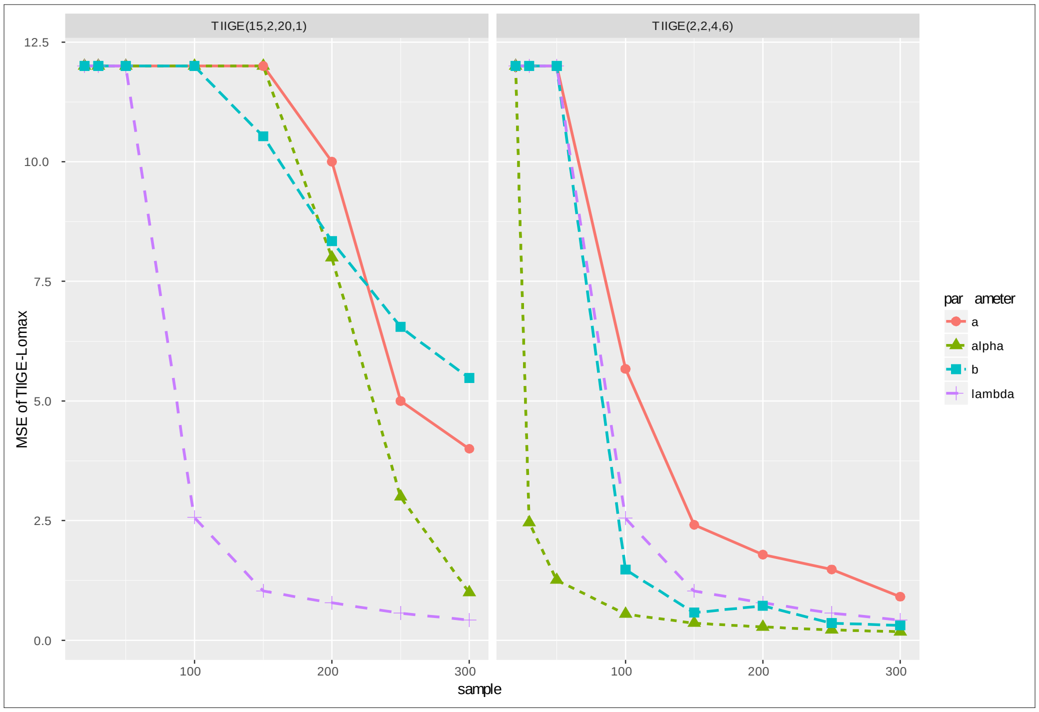


Figure 2: The MSE of different parameters of TIIGE-LO(15,2,20,1) and TIIGE-LO(2,2,4,6) versus n = 20, 30, 50, 100, 150, 200, 250, and 300.

Table 1: Any information that is needed based on the output interpretation

Table 1: Any information that is needed based on the output interpretation

Actual values	n	Bias(α, λ, a, b)	MSE(α, λ, a, b)	CI(α, λ, a, b)
20,1, 15,2	20	15.87, 14.53, 45.05, 14.99	878.97, 4210.91, 9300.80, 2544.84	-10.34, 72.08, -87.55, 120.61, 4.95, 31.72, -10.72, 21.16
	30	4.45, 3.15, 10.61, 3.77	138.38, 363.99, 866.13, 251.13	1.53, 37.37, -25.81, 38.11, 9.00, 30.18, -9.42, 14.97
	50	2.71, 0.99, 4.62, 1.83	71.02, 60.27, 191.55, 65.79	4.39, 30.84, -9.69, 13.66, 9.77, 28.44, -9.29, 15.69
	100	1.36, 0.21, 1.66, 0.78	31.21, 2.57, 36.24, 17.00	7.45, 25.28, -0.41, 4.82, 14.98, 29.86, -6.54, 6.86
	150	0.85, 0.12, 1.28, 0.60	20.15, 1.03, 22.11, 10.53	8.60, 23.11, 0.47, 3.78, 13.25, 27.38, -3.98, 7.24
	200	0.56, 0.12, 0.73, 0.40	13.85, 0.78, 17.24, 8.34	9.07, 22.04, 0.68, 3.56, 16.63, 23.27, -3.09, 6.35
	250	0.43, 0.10, 0.71, 0.26	12.51, 0.57, 13.53, 6.55	9.65, 21.21, 0.86, 3.33, 16.17, 24.75, -2.04, 3.33
	300	0.42, 0.06, 0.69, 0.24	9.97, 0.42, 10.81, 5.48	10.28, 20.57, 0.99, 3.13, 19.85, 29.50, -0.91, 2.42
20,1, 2,3	20	2.12, 13.87, 35.86, 9.08	15.63, 4598.61, 9208.95, 2306.13	-1.38, 9.61, -90.61, 126.34, -11.03, 15.72, -5.72, 26.16
	30	0.59, 4.18, 8.80, 2.44	2.36, 776.29, 1554.98, 389.30	0.20, 4.98, -20.15, 51.51, -6.40, 14.18, -4.42, 19.97
	50	0.36, 1.20, 2.75, 0.81	1.26, 112.08, 227.27, 57.26	0.61, 4.11, -14.18, 20.58, -6.23, 12.44, -4.29, 20.69
	100	0.18, 0.21, 0.54, 0.22	0.55, 2.55, 5.07, 1.48	0.99, 3.37, -0.40, 4.81, -1.02, 13.86, -1.54, 11.88
	150	0.11, 0.12, 0.30, 0.09	0.36, 1.03, 2.41, 0.58	1.15, 3.08, 0.47, 3.78, -2.75, 11.38, 1.02, 12.24
	200	0.07, 0.12, 0.25, 0.22	0.28, 0.78, 1.78, 0.72	1.21, 2.94, 0.68, 3.56, 0.63, 7.27, 1.91, 11.35
	250	0.06, 0.10, 0.33, 0.02	0.22, 0.57, 1.48, 0.38	1.29, 2.83, 0.86, 3.33, 0.17, 8.73, 2.96, 8.33
	300	0.06, 0.06, 0.24, 0.04	0.18, 0.42, 0.91, 0.31	1.37, 2.74, 0.99, 3.13, 3.95, 4.50, 4.09, 7.42

5. Special TIIGE models

In this section, we provide two special models of the TIIGE family. These special models generalize some wellknown distributions reported in the literature. They correspond to the baseline Lomax (LO) and Lindley (L) distributions and illustrate the flexibility of the new family.

5.1 The TIIGE-Lomax distribution

Consider the pdf $g(x)=\frac{a}{b}\left(1+\frac{x}{b}\right)^{-(a+1)}$ and cdf $G(x)=1-\left(1+\frac{x}{b}\right)^{-a}$ of the LO distribution with scale $b>0$ and shape $a>0$ parameters. Inserting these functions in (4), the pdf of the TIIGE-LO model (for $x>0$) is given by

$$f(x)=\lambda\alpha\frac{a}{b}\left(1+\frac{x}{b}\right)^{-(a+1)}\left(1+\frac{x}{b}\right)^{a(\alpha+1)}\exp\left(\lambda\left(1-\left(1+\frac{x}{b}\right)^{a\alpha}\right)\right)$$

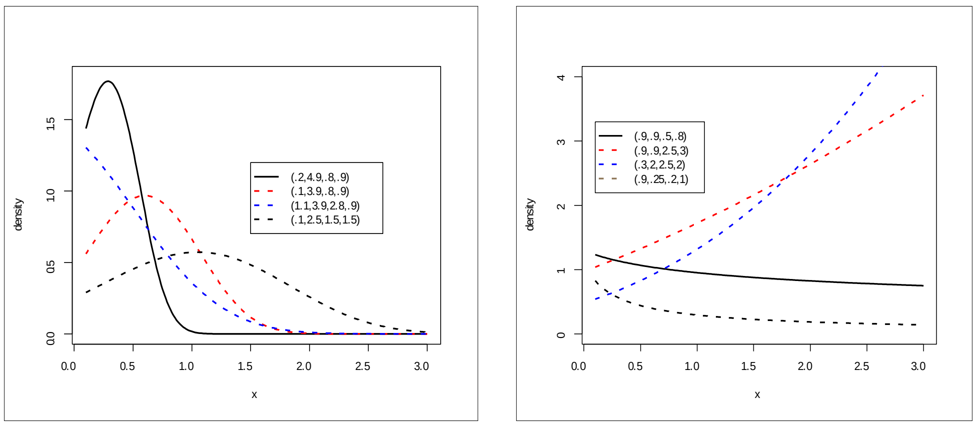


Figure 3: TIIGE-LO model: pdf (left), hrf (right).

5.2 The TIIGE-Lindley distribution

Consider the Lindley distribution with parameter θ and the pdf and cdf (for $x>0$) is given by

$$g(x)=\frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x),$$

and

$$G(x)=1-\frac{\exp(-\theta x)(1+\theta+\theta x)}{\theta+1}.$$

Inserting these expressions in (4) gives the TIIGE-L density function

$$f(x)=\lambda\alpha\frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x)\left(\frac{\exp(-\theta x)(1+\theta+\theta x)}{\theta+1}\right)^{-(\alpha+1)}$$

$$\exp\left(\lambda\left(1-\left(\frac{\exp(-\theta x)(1+\theta+\theta x)}{\theta+1}\right)^{-\alpha}\right)\right).$$

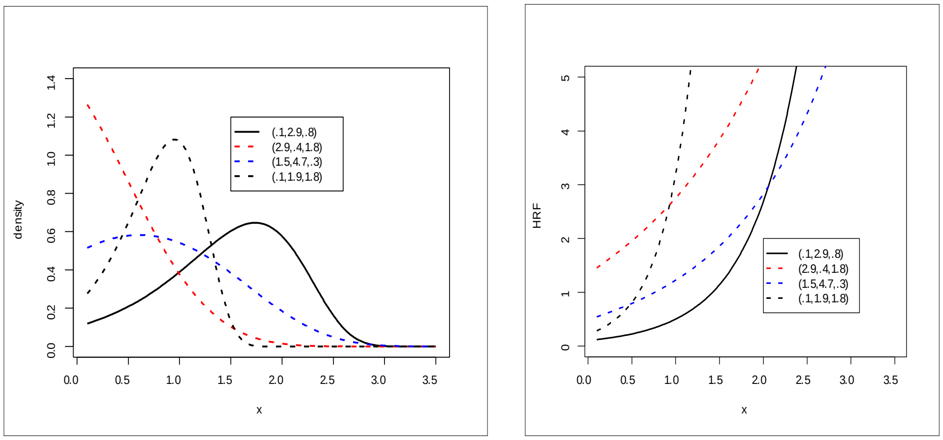


Figure 4: T**II**GE-L distribution: pdf (left), hrf (right).

6. Data analysis

In this section, we use two real data sets to compare the fits of the T**II**GE-G family with others commonly used lifetime family of distributions. In each case, the parameters of models are estimated by maximum likelihood (Section 4) using the optim function in R program. First, we describe the data sets and give the MLEs (the corresponding standard errors) of the model parameters and the values of the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn information criterion (HQIC) and Anderson-Darling (A^*) statistics. The lower the values of these criteria shows the better fitted model to dataset. Its worth to mention that over-parametrization is penalized in these criteria. Finally, we provide the histograms of the data sets to have a visual comparison of the fitted density functions.

Example 1 (Failure times of Aircraft Windshield): This data set presented in Murthy et al. (2004) and used by some reserchers for example Elbatal et al. (2016). This data set present failure times for a particular windshield model including 85 observations that are classified as failed times of windshields. This data set is given by 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924,1.281,2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480,2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615,2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

First, we describe the descriptive statistics of the data in Table 2. In Table 3, This data set is fitted by four parameter T**II**GE-LO distribution. The rival models for this data set are 4 parameter models Beta Lomax (Mead, 2016), Kumaraswamy Lomax (Elbatal and Kareem, 2014) and 5 parameter models McDonald Lomax (Lemonte and Cordeiro, 2011). Table 4 present Good ness of fit critera for fitted models. This table reveals that the T**II**GE-LO model gives a better fit to this data than the other models.

Table 2: Descriptive statistics of first data set.

Mean	Median	Variance	Skewness	Kurtosis	Min	Max	n
2.563	2.385	1.239	0.086	2.365	0.040	4.663	85

Table 3: Parameters estimates and corresponding standard errors for first data set.

Model	Estimates (Standard Error)
TIIGE-LO	0.014, 3.790, 1.048, 1.559
(λ, α, a, b)	(0.013), (18.809), (5.204), (1.328)
B-LO	3.529, 64899.409, 0.432, 20391.612
(α, β, a, b)	(0.517), (8.43×10^2), (0.088), (76.881)
Kw-LO	2.409, 39891.205, 1.519, 349.726
(α, β, a, b)	(0.215), (16.7×10^2), (1.248), (265.123)
Mc-LO	0.301, 239.685, 47.202, 334.323, 6.698
(α, β, a, b, c)	(0.094), (273.996), (57.370), (131.775), (1.331)

Table 4: Goodness of fit statistics for first data set.

Model	Goodness of fit criteria				
	AIC	*BIC*	*HQIC*	*CAIC*	*A* 🖓
TIIGE-LO	264.036	273.807	267.966	264.536	0.552
B-LO	284.794	294.564	288.724	285.294	1.367
Kw-LO	270.779	280.550	274.709	271.279	0.593
Mc-LO	269.231	281.444	274.143	269.990	0.609

Example 2 (Glass Fiber): The second data set is given by Smith and Naylor (1987) on he strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England . The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64,1.68, 1.73, 1.81, 2 ,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01,0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48,1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89. The summary statistics about this data set present in Table 5. For this data set, we use T**II**GE-L distribution and rival models are with Kw-L (Cakmakyapan and Kadilar, 2014), Beta-L (MirMostafaee et al., 2015) and Beta Exponentiated-L (Rodrigues et al., 2015). Table 6 presents that the estimates and standard errors of models parameters. Table 7 present Goodness of fit critera for fitted models. This table shows that the T**II**GE-L model gives a better fit to this data than the other distributions.

Table 5: Descriptive statistics of second data set.

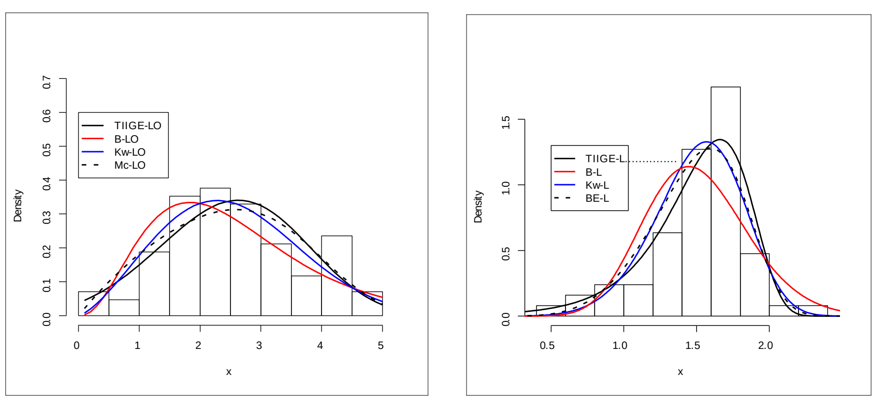
Mean	Median	Variance	Skewness	Kurtosis	Min	Max	n
1.507	1.590	0.105	-0.899	3.923	0.550	2.440	63

Table 6: Parameters estimates and corresponding standard errors for second data set.

Model	Estimates (Standard Error)
TIIGE-L	0.004, 3.473, 1.341
$(\lambda, \alpha, \theta)$	(0.002), (10.731), (3.277)
B-L	11.291, 80.550, 0.271
(α, β, θ)	(2.193), (48.606), (0.101)
Kw-L	5.391, 1828.367, 0.415
(α, β, θ)	(0.881), (2335.611), (0.120)
BE-L	0.391, 162.955, 1.096, 17.138
$(\alpha, \beta, \theta, c)$	(0.224), (242.412), (0.477), (13.638)

Table 7: Formal goodness of fit statistics for second dataset

Model	Goodness of fit criteria				
	AIC	*BIC*	*HQIC*	*CAIC*	A^*
TIIGE-L	36.169	42.599	38.698	36.576	0.841
B-L	50.310	56.740	52.839	50.717	2.783
Kw-L	36.839	43.268	39.368	37.246	1.363
BE-L	38.104	46.676	41.475	38.793	1.259



(a) (b)

Figure 5: Fitted pdfs on histogram of first and second data sets.

7. Conclusions

A new class of distributions called the Type **II** General Exponential class is introduced and studied. We provide a comprehensive treatment of some of its mathematical properties including ordinary and incomplete moments, generating function, asymptotics, order statistics and the QS order, moment of residual life and reversed residual life. We estimate the model parameters by the maximum likelihood method. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study. The potentiality of the proposed models is illustrated by means of three real data sets.

References

1. Afify A. Z., Alizadeh, M., Yousof, H. M., Aryal, G. and Ahmad, M. (2016a). The transmuted geometric-G family of distributions: theory and applications. Pak. J. Statist., forthcoming.
2. Afify A. Z., Cordeiro, G. M., Yousof, H. M., Alzaatreh, A. and Nofal, Z. M. (2016b). The Kumaraswamy transmuted-G family of distributions: properties and applications. J. Data Sci., forthcoming.
3. Afify, A.Z., Yousof, H. M. and Nadarajah, S. (2016c). The beta transmuted-H family of distributions: properties and applications. Stasistics and its Inference, forthcoming.
4. Alizadeh, M., Cordeiro, G. M., Nascimento, A. D. C. Lima M. D. S. and Ortega, E.M.M. (2016a). Odd-Burr generalized family of distributions with some applications. Journal of Statistical Computation and Simulation, 83, 326-339.
5. Alizadeh, M., Yousof, H. M., Afify A. Z., Cordeiro, G. M., and Mansoor, M. (2016b). The complementary generalized transmuted poisson-G family. Austrian Journal of Statistics, forthcoming.
6. Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, **71**, 63-79.
7. Cakmakyapan, S. and Kadilar, G.O. (2014). A New Customer Lifetime Duration Distribution:The Kumaraswamy Lindley Distribution. International Journal of Trade, Economics and Finance, **5**, 441-444.
8. Cook, R. D., Weisberg, S. (1994), An Introduction to Regression Graphics. John Wiley.
9. Cordeiro, G. M., Alizadeh, M. and Diniz Marinho, P. R. (2016a). The type I half-logistic family of distributions. Journal of Statistical Computation and Simulation, **86**, 707-728.
10. Cordeiro, G. M., Alizadeh, M., Ortega, E. M. and Serrano, L. H. V.(2016b). The Zografos-Balakrishnan odd log-logistic family of distributions: Properties and Applications. Hacet. J. Math. Stat., forthcoming.
11. Cordeiro, G. M., Alizadeh, M., Ozel, G., Hosseini, B., Ortega, E. M. M. and Altun, E. (2016c). The generalized odd log-logistic family of distributions: properties, regression models and applications, Journal of Statistical Computation and Simulation, http://dx.doi.org/10.1080/00949655.2016.1238088.
12. Cordeiro, G. M., Alizadeh, M., Tahir, M. H., Mansoor, M., Bourguignon, M. and and Hamedani G.G. (2016d). The beta dd log-logistic generalized family of distributions, Hacettepe Journal of Mathematics and Statistics, Forthcoming.
13. Elbatal, I. and Kareem, A. (2014). Statistical Properties of Kumaraswamy Exponentiated Lomax Distribution. Journal of Modern Mathematics and Statistics, **8**, 1-7.
14. Elbatal, I. and Mansour, M. M. (2016). The Additive Weibull-Geometric Distribution: Theory and Applications. Journal of Statistical Theory and Applications, **15**, 125-141.

15. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 75-84.
16. Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments, Statistics: A Journal of Theoretical and Applied Statistics, **21**, 613-618.
17. Gupta, P. K., Singh, B. (2013). International Journal of System Assurance Engineering and Management. International Journal of System Assurance Engineering and Management, **4**: 378-385.
18. Gupta, R. C., Gupta, P. L. and Gupta, R. D. (1998). Modeling failure time data by Lehmann alternatives. Commun. Stat. Theory Methods, **27**, 887-904.
19. Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions. Australian and New Zealand Journal of Statistics, **41**, 173–188.
20. Hamedani, G. G. (2013). On certain generalized gamma convolution distributions **II**, Technical Report, No. 484, MSCS, Marquette University.
21. Korkmaz, M. C. and Genc, A. I. (2016). A new generalized two-sided class of distributions with an emphasis on two-sided generalized normal distribution. Communications in Statistics
22. Kotz, S., Lai, C. D. and Xie, M. (2003). On the effect of redundancy for systems with dependent components." IIE Trans, **35**, 1103–1110.
23. Kotz, S. and Seier, E. (2007). Kurtosis of the Topp Leone distributions. Interstat, 1-15.
24. Lee, C., Famoye, F. and Olumolade, O. (2007).Beta-Weibull Distribution: Some Properties and Applications to Censored Data. Journal of Modern Applied Statistical Methods, **6**, 173-186.
25. Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. (2006). Theory and Applications of Fractional Difierential Equations. Elsevier, Amsterdam.
26. Kumaraswamy, P. (1980). A generalized probability density function for double bounded random processes. Journal of Hydrology, **46**, 79–88.
27. Lemonte, A. J. and Cordeiro, G. M. (2013). An extended Lomax distribution. Statistics: A Journal of Theoretical and Applied Statistics , **47**, 800-816. Lemonte, A.J., Barreto-Souza, W. and Cordeiro, G. M. (1013). The exponentiated Kumaraswamy distribution and its log-transform. Brazilian Journal of Probability and Statistics, **27**, 31-53.
28. Marshall, A. W. and Olkin, I. (1997). A new methods for adding a parameter to a family of distributions with application to the Exponential and Weibull families. Biometrika, **84**, 641-652.
29. Mathai, A. M. and Saxena, R. K. (1978). The H-Function with Applications in Statistics and Other Disciplines. John Wiley and Sons, New York.
30. Mead, M.E. (2016). On Five-Parameter Lomax Distribution:Properties and Applications. Pakistan Journal of Statistics and Operation Research, **12**, 185-199.
31. Merovci, F., Alizadeh, M., Yousof, H. M. and Hamedani G. G. (2016). The exponentiated transmuted-G family of distributions: theory and applications. Commun. Stat. Theory Methods, Forthcoming.
32. MirMostafaee, S. M. T. K., Mahdizadeh, M. and Nadarajah, S. (2015). The beta Lindley distribution. Journal of Data Science, **13**, 603-626.

33. Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analysing bathtub failure rate data. IEEE Transactions on Reliability, **42**, 299–302.
34. Mudholkar, G.S., Srivastava, D.K. and Freimer, M. (1995). The exponentiated Weibull family: a reanalysis of the bus-motor-failure data. Technometrics, **37**, 436–445.
35. Murthy, D. N. P., Xi, M. and Jiangs, R. (2004). Weibull models, Wiley, Hoboken.
36. Nadarajah, S. (2005). The exponentiated Gumbel distribution with climate application. Environmetrics, **17**, 13–23.
37. Nadarajah, S. and Gupta, A.K. (2007). The exponentiated gamma distribution with application to drought data. Calcutta Statistical Association Bulletin, **59**, 29–54.
38. Nadarajah, S. and Kotz, S. (2003). Moments of some J-shaped distributions. Journal of Applied Statistics, **30**, 311-317.
39. Nadarajah, S. and Kotz, S. (2006). The exponentiated-type distributions. Acta Applicandae Mathematicae, **92**, 97–111.
40. Nadarajah, S., Cordeiro, G. M. and Ortega, E. M. M. (2013). The exponentiated Weibull distribution: A survey, Statistical Papers, **54**, 839–877.
41. Nadarajah, S., Cordeiro, G. M. and Ortega, E. M. (2015). The Zografos Balakrishnan-G family of distributions: Mathematical properties and applications. Communications in Statistics-Theory and Methods, **44**, 186-215.
42. Nofal, Z. M., Afify, A. Z., Yousof, H. M. and Cordeiro, G. M. (2015). The generalized transmuted-G family of distributions. Commun. Stat. Theory Methods, forthcoming.
43. Nichols, M. D. and Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles.Quality and Reliability Engineering International, **22**, 141-151.
44. Rodrigues, J. A., Silva, A. P. C. M., Hamedani, G. G.(2015). The beta exponentiated Lindley distribution. Journal of Statistical Theory and Applications, **14**, 60-75.
45. Rezaei, S., Sadr, B.B., Alizadeh, M. and Nadarajah, S. (2016). Topp-Leone generated family of distributions: Properties and applications, Communications in Statistics-Theory and Methods, forthcoming.
46. Shanker, R. (2015). Shanker Distribution and Its Applications. International Journal of Statistics and Applications, **5**, 338-348.
47. Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. Research report.
48. Shirke, D. T and Kakade, C.S. (2006). On exponentiated lognormal distribution. International Journal of Agricultural and Statistical Sciences, **2**, 319–326.
49. Sindhu, T. N., Saleem, M. and Aslam, M. (2013). Bayesian estimation for Topp Leone distribution under trimmed samples. Journal of Basic and Applied Scientific Research, **3**, 347-360.
50. Silva, F. G., Percontini, A., de Brito, E., Ramos, M.W., Venancio, R. and Cordeiro, G. M. (2016). The odd Lindley-G family of distributions, Austrian Journal of Statistics, **VV**, 1–xx.
51. Smith, R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Journal of Mechanical Science and Technology, **36**, 358-369

52. Srivastava, H. M., Gupta, K. C. and Goyal, S. P. (1982). The H-Functions of One and Two Variables with Applications. South Asian Publishers, New Delhi.
53. Tahir, M. H., Cordeiro, G.M., Alzaatreh, A., Mansoor, M. and Zubair, M. (2016a). The logistic-X family of distributions and its applications. Commun. Stat. Theory Methods, forthcoming.
54. Tahir, M. H., Zubair, M., Mansoor, M., Cordeiro, G.M., Alizadeh, M. amd Hamedani, G.G. (2016b). A new Weibull-G family of distributions. Hacet. J. Math. Stat., forthcoming.
55. Weibull, W. (1951). A statistical distribution function of wide applicability. J. Appl. Mech. Trans, **18**, 293–297.
56. Wright, E. M. (1935). The asymptotic expansion of the generalized hypergeometric function. Journal of the London Mathematical Society, **10**, 286-293.
57. Yousof, H. M., Afify, A. Z., Hamedani, G. G. and Aryal, G. (2016). the Burr X generator of distributions for lifetime data. Journal of Statistical Theory and Applications, forthcoming.

Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty, e = \infty$ mightaswellbeallowed$)$. Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}[q_2(X)|X \geq x] = \mathbf{E}[q_1(X)|X \geq x]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1,\ q_2 \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η , particularly

$$F(x) = \int_a^x C \left|\frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)}\right| \exp\big(-s(u)\big)du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$. We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel 1990), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions $q_{1,n}$, $q_{2,n}$ and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $q_{1,n} \to q_1$, $q_{2,n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F . Under the condition that $q_{1,n}(X)$ and $q_{2,n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η , where

$$\eta(x) = \frac{E[q_2(X)|X \geq x]}{E[q_1(X)|X \geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1 , q_2 and η , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glänzel and Hamedani (2001).

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take $q_1(x) \equiv 1$, as we did in Proposition 3.1, which reduces the condition of Theorem 1 to $\mathbf{E}[q_2(X)|X \geq x] = \eta(x), \quad x \in H$. We, however, believe that employing three functions q_1 , q_2 and η will enhance the domain of applicability of Theorem 1.