# Length-biased Weighted Maxwell Distribution

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## Abstract

The concept of length-biased distribution can be employed in development of proper models for life-time data. In this paper, we develop the length-biased form of Weighted Maxwell distribution (WMD). We study the statistical properties of the derived distribution including moments, moment generating function, hazard rate, reverse hazard rate, Shannon entropy and estimation of parameters.

**Keywords**: Weighted Maxwell distribution, Moments, moment generating function, Hazard rate, Shannon entropy.

## Introduction

The weighted distributions are used as a tool in selection of appropriate models for observed data, especially when samples are drawn without a proper frame. It has been employed in wide variety applications in reliability and survival analysis, analysis of family data, meta-analysis, ecology, and forestry.

When an investigator records an observation by nature according to a certain stochastic model the recorded observation will not have the original distribution unless every observation is given an equal chance of being recorded. For example, suppose that the original observation X has f(x) as the pdf, and that the probability of recording the observation x is $0 \le w(x) \le 1$, then the pdf of weighted random variable $X_w$ is given by:

$$f_w(x) = \frac{w(x)f(x)}{\mu} \qquad , x > 0 \tag{1}$$

where $\mu = E\big(w(x)\big) < \infty$, is the normalizing factor obtained to make the total probability equal to unity.

When the weight function depends on the lengths of units of interest i.e. w(x)=x, the resulting distribution is called length-biased. In this case, the pdf of a length-biased (rv) $X_L$ is defined as

$$f_L(x) = \frac{x\, f_W(x)}{\mu} \tag{2}$$

where $\mu = E\big(w(x)\big) < \infty$.

Length-biased distributions are a special case of the more general form known as weighted distributions.

Weighted distribution provides an approach to dealing with model specification and data interpretation problems. Fisher (1934) and Rao (1965) introduced and unified the concept of weighted distribution. When the probability of observing a positive-valued random variable is proportional to the value of the variable the resultant is length-biased distribution. A table for some basic distributions and their length-biased forms is given by Patil and Rao (1978) such as Lognormal, Gamma, Pareto, Beta distribution. Khatree (1989) presented a useful result by giving a relationship between the original random variable X and its length-biased version Y when X is either Inverse Gaussian or Gamma distribution. Das et al. (2011) developed the length biased distribution (LBD) of weighted Inverse Gaussian distribution (WIGD). Das and Roy (2011) developed the length-biased form of the Weighted Generalized Rayleigh distribution (WGRD) known as length-biased Weighted Generalized Rayleigh distribution. Al-kadim and Hussein (2014) derived length-biased Weighted Exponential and Rayleigh distributions and studied their properties.

The Maxwell distribution applies to ideal gases close to thermodynamic equilibrium with negligible quantum effects and at non-relativistic speeds. It forms the basis of the kinetic theory of gases, which provides a simplified explanation of many fundamental gaseous properties, including pressure and diffusion. In this paper we explore the statistical properties of Length biased Weighted Maxwell distribution (LBWMD). We derive expression for moments, moment generating function, hazard rate function. Also estimate its parameter using method of estimation.

The following identities are used in this paper

**Identity 1**: From Gradshteyn and Ryzhik (1965), Equation (3.462.1), Page 382.
For Re β>0, Re ν>0, we have

$$\int_0^\infty x^{\nu-1} e^{-\beta.x^2-\gamma.x} dx = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$

**Identity 2**: From Gradshteyn and Ryzhik (1965), Equation (3.381.4), Page 364.
For Re μ>0, Re ν>0, we have

$$\int_0^\infty e^{-\mu.x} x^{\nu-1} dx = \frac{\Gamma(\nu)}{\mu^\nu}$$

**Identity 3**: From Prudnikov, Brychkov and Marichev (2005), Equation (2.6.21.2), Page 527.

For Re α, μ, Re p >0, we have

$$\int_0^\infty x^{\alpha-1} e^{-px^\mu} \ln x dx = \mu^{-2} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) \cdot \left[\psi\left(\frac{\alpha}{\mu}\right) - \ln p\right]$$

**Identity 4**: From Gradshteyn and Ryzhik (1965), Equation (3.381.1), Page 364.
For Re ν>0, we have

$$\int_0^u e^{-\mu.x} x^{\nu-1} dx = \frac{\gamma(\nu, \mu.u)}{\mu^\nu}$$

## 1. Length-biased Weighted Maxwell Distribution

The probability density function (pdf) of Maxwell distribution is given by:

$$f(x;a)=\frac{\sqrt{2}a^{3/2}.x^2.\exp\left(-\frac{ax^2}{2}\right)}{\sqrt{\pi}} \qquad x>0,\ a>0 \qquad (3)$$

The pdf of Weighted Maxwell distribution with weighted function $w(x;t)=e^{tx^2/2}$ and for x > 0; a > 0; and a>t is given by (refer Joshi and Modi (2012)):

$$f_w(x;a,t)=\frac{\sqrt{2}(a-t)^{3/2}.x^2.\exp\left(-\frac{x^2}{2}(a-t)\right)}{\sqrt{\pi}} \qquad x>0,\ a>0 \text{ and } a>t \qquad (4)$$

Note that

$$\lim_{x\to 0} f_w(x;a,t)=0 \quad and \quad \lim_{x\to\infty} f_w(x;a,t)=0$$

And the cdf corresponding to the weighted Maxwell distribution is given by:

$$F_w(x;a,t)=\int_0^x f_w(y;a,t)dy$$

$$F_w(x;a,t)=\frac{2.\gamma\left(\frac{3}{2},\left(\frac{a-t}{2}\right)x^2\right)}{\sqrt{\pi}} \qquad (5)$$

Thus we can derive pdf of Length biased Weighted Maxwell distribution (LBWMD) by using (2) and (4) as:

$$f_L(x)=\frac{x.f_w(x)}{\int_0^\infty x.f_w(x)dx} \qquad ,\ x>0$$

$$f_L(x;a,t)=\frac{8.x^3.\exp\left(-\frac{x^2}{2}(a-t)\right)}{(a-t)^2} \qquad (6)$$

And using identity 4, its corresponding cdf is given by:

$$F_L(x;a,t)=\frac{16.\gamma\left(2,\left(\frac{a-t}{2}\right)x^2\right)}{(a-t)^4} \qquad (7)$$

## 2. Nature Of The Distribution

The pdf of Length biased Weighted Maxwell distribution (LBWMD) given by (6) is:

$$f_L(x;a,t) = \frac{8.x^3.\exp\left(-\frac{x^2}{2}(a-t)\right)}{(a-t)^2} \qquad a>0 \text{ and } a>t,\ x\geq 0$$

Its first derivative is given by:

$$f_L^{'}(x;a,t) = \frac{f_L(x;a,t)}{x}.\left[3 - x^2(a-t)\right] \tag{8}$$

Thus standard calculations based on this derivative shows that f(x) exhibits positive mode at $x = \sqrt{\frac{3}{(a-t)}}$ with $f_L(0) = 0$, $f_L(\infty) = 0$.

**Graph of p.d.f. of LBWMD for t=0.5 with different values of a= 5,10,15 (left) and for t=1 with different values of a= 5,10,15 (right).**

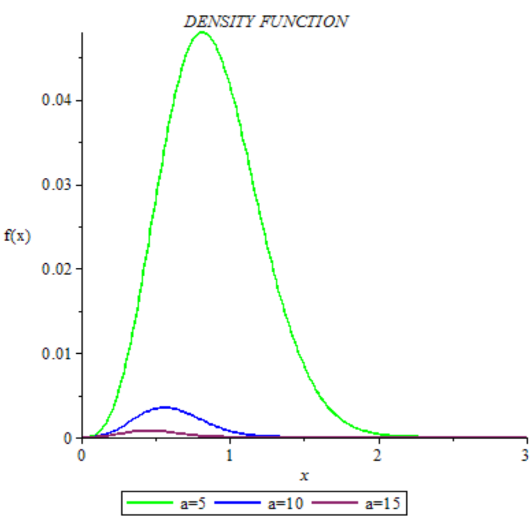


Fig. 1.1

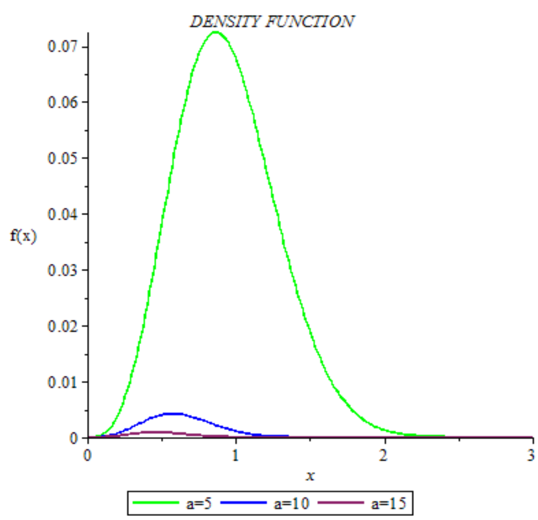


Fig. 1.2

## 3. Hazard Rate Function

The hazard rate function of LBWMD defined in (6) can be calculated as:

$$h_L(x;a,t) = \frac{8(a-t)^2.x^3.\exp\left(-\frac{x^2}{2}(a-t)\right)}{\left[(a-t)^4 - 16.\gamma\left(2,\left(\frac{a-t}{2}\right)x^2\right)\right]} \tag{9}$$

**Graph of hazard rate function of LBWMD for t=0.5 with different values of a= 1,2,3 (left) and for t=1 with different values of a= 2,3,4 (right).**

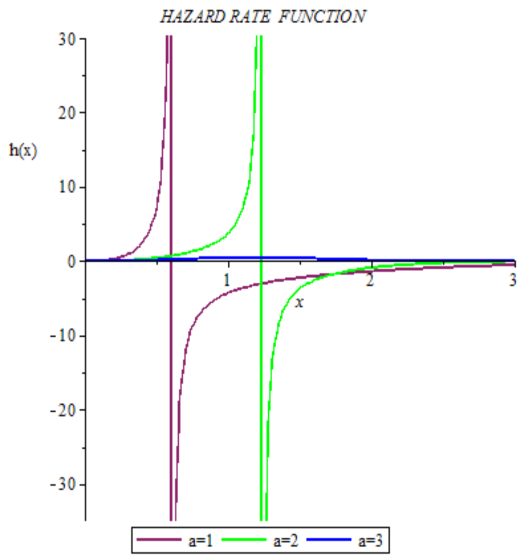


Fig. 2.1

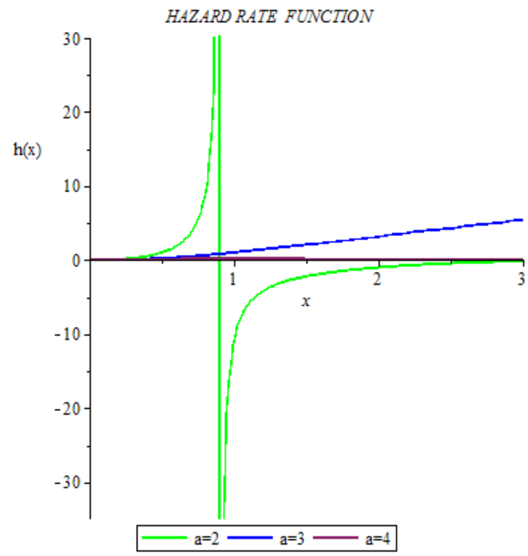


Fig. 2.2

## 4. Reverse Hazard Rate Function

The reverse hazard rate function of LBWMD defined in (6) can be calculated as:

$$\tau_L(x;a,t) = \frac{f_L(x)}{F_L(x)}$$

$$= \frac{(a-t)^2 . x^3 . \exp\left(-\frac{x^2}{2}(a-t)\right)}{2.\gamma\left(2,\left(\frac{a-t}{2}\right)x^2\right)} \tag{10}$$

## 5. Moment Generating Function

The corresponding mgf of LBWMD defined in (6) is given by:

$$M_L(p) = \int_0^\infty e^{px} . f_L(x;a,t)dx$$

$$= \frac{8}{(a-t)^2} \int_0^\infty x^3 . \exp\left(px - \frac{x^2}{2}(a-t)\right)dx$$

Thus applying identity 1 and solving, we get

$$M_L(p) = \frac{48}{(a-t)^4} . \exp\left\{\frac{p^2}{4(a-t)}\right\} . D_{-4}\left(\frac{-p}{\sqrt{(a-t)}}\right) \tag{11}$$

## 6. Moments

If a random variable X has the pdf given by equation (6), then the corresponding r<sup>th</sup> moment is given by:

$$E(x^r) = \int_0^\infty x^r . f_L(x;a,t)dx$$

Thus, $E(x^r) = \frac{8}{(a-t)^2} \int_0^\infty x^{3+r}.\exp\left(-\frac{x^2}{2}(a-t)\right)dx$

Thus applying identity 2 and solving, we get

$$\therefore E(x^r) = \frac{8}{(a-t)^2} \left(\frac{2}{a-t}\right)^{(r+2)} \Gamma(2+r) \quad (12)$$

## 7. Entropy

Entropy of a random variable is a measure of variation of the uncertainty. The Shannon entropy (1948) defined by:

$$E[-\ln f_L(X)] = -\int_0^\infty \ln f_L(x;a,t).f_L(x;a,t).dx$$

Using equation (6), we get

$$E[-\ln f_L(X)] = -\int_0^\infty \ln\left[\frac{8.x^3.\exp\left(-\frac{x^2}{2}(a-t)\right)}{(a-t)^2}\right].\left[\frac{8.x^3.\exp\left(-\frac{x^2}{2}(a-t)\right)}{(a-t)^2}\right]dx$$

$$= \int_0^\infty \left[\frac{4x^5 \exp\left(-\frac{x^2(a-t)}{2}\right)}{(a-t)} - \frac{24.x^3 \exp\left(-\frac{x^2(a-t)}{2}\right).\ln x}{(a-t)^2}\right.$$

$$\left. + \frac{(16\ln(a-t) - 8\ln 8).x^3 \exp\left(-\frac{x^2(a-t)}{2}\right)}{(a-t)^2}\right]dx$$

Thus applying identity 2, identity 3 and solving, we get

$$E[-\ln f_L(X)] = \frac{32}{(a-t)^4} + \frac{32\ln(a-t) - 16\ln 8}{(a-t)^4} - \frac{24}{(a-t)^4}\left[\psi(2) - \ln\left(\frac{a-t}{2}\right)\right] \quad (13)$$

## 8. Estimation

Let X is a random variable having the pdf defined as:

$$f_L(x;a,t) = \frac{8.x^3.\exp\left(-\frac{x^2}{2}(a-t)\right)}{(a-t)^2}$$

Then its log-likelihood function can be written as:

$$L(a) = n\ln 8 - \sum_{i=1}^{n} \frac{x_i^{\,2}}{2}(a-t) + 3\sum_{i=1}^{n} \ln x_i - 2n\ln(a-t)$$

Thus the non-linear normal equation is given by:

$$\frac{\partial L}{\partial a} = -\sum_{i=1}^{n} \frac{x_i^{\,2}}{2} - \frac{2n}{(a-t)}$$

Equating $\frac{\partial L}{\partial a} = 0$, we obtain the estimated value of parameter 'a' as:

$$\hat{a} = -\frac{4n}{\sum_{i=1}^{n} x_i^{\,2}} + t \tag{14}$$

## 9. Conclusion

In this paper, we derive the Length biased Weighted Maxwell distribution (LBWMD) to increase its application area. We also studied its statistical properties like mean, moments, moment generating functions, hazard rate function, etc. We have plotted graphs for its density function and hazard rate function at t=0.5, 1 for different values for their parameters. The graph of density function depicts that distribution is positively skewed, more peaked for small values of parameter 'a'.

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