Optimization of Fuzzy Production Inventory Models for Crisp or Fuzzy Production Time

Harun Öztürk
Süleyman Demirel University, Turkey
harunozturk@sdu.edu.tr

Abstract

Managing the inventories is very important task for companies in the manufacturing industry. In this paper, we first present a mathematical model to determine an optimal production time for a single-stage production inventory problem with rework process. Next, we consider the production inventory problem in fuzzy environment by applying two types of fuzzy numbers, which are trapezoidal and triangular. Fuzzy total inventory cost functions are derived for both production inventory models with crisp production time period and fuzzy production time period, respectively, and defuzzified by using graded mean integration representation method. To illustrate the results of developed models, numerical examples are provided, and sensitivity analysis is carried out to discuss the effects of the fuzziness in the components over the production time and the total inventory cost.

Keywords: Inventory; Single-stage; Production time; Lagrangean; Optimization; Fuzzy set theory.

1. Introduction

At the beginning of the 1990s, the Economic Order Quantity (EOQ) and the Economic Production Quantity (EPQ) models have been one of the most important subjects of production and operations management areas. Although there has been a significant effort by academicians and researchers to provide applicability of the inventory models, these models ignore many factors faced with real-world problems. For example, one of the basic assumptions of these models is that 100% of items produced or received are of perfect quality. However, in most of production processes, it is unavoidable that the items ordered or produced have defects. Recently, these models have been extended in many directions by relaxing the assumption. Salameh and Jaber (2000) extended the classical EOQ model by assuming that the ordered lot contains a random proportion of defective items and after the inspection of the whole lot, the identified defective items are sold in a single batch at a lower price. Chen (2003) considered the classical EOQ model under random demand. Chen used the cycle length T as decision variable. Jamal et al. (2004) developed two production inventory models to determine the optimal production quantity in a single-stage production system where rework is done under two different cases to minimize the total inventory cost. In the first case, the defective items produced regular production process is reworked within the same cycle. In the second case, the defective items are accumulated until N cycles are completed, and then they are reworked. Later, Cárdenas-Barrón (2009) extended Jamal et al.'s (2004) model, where defective items are reworked within the same cycle, considering planned backorders. An extensive review of the models that deal with the classical inventory models can be found in Andriolo et al. (2014).

In present conditions, companies need to apply different inventory models because of changing its management strategies, production types, financial requirements and other factors. This may increase the degree of uncertainty that decision makers take into account when planning. However, there are many difficulties to know the exact value of input parameters including vagueness and imprecision for the development of a mathematical modeling related to the real-world problems. In the classical inventory models, in order to determine the optimal production and order quantities, both demand and cost parameters are assumed to be fix values. However, in the real-world problems, all of them probably will have little changes. In literature, most of inventory models assume that input parameters and decision variables are described as crisp values or having crisp statistical distributions to reflect these uncertainties which have a significant importance for decision makers.

Uncertainty that has a quantifiable imprecision can arise from observation and measurement. The probability theory could address this kind of imprecision. However, there are additional sources of uncertainty from incomplete information and data, lack of knowledge, vagueness and ambiguities mentioned in Booker and Ross (2011). Fuzzy set theory was introduced by Zadeh in 1965to represent this kind of imprecision. Since then, it has been successfully applied in industry, and there are many different inventory problems related to the fuzzy set theory in production management. Park (1987) fuzzified ordering cost and holding cost into trapezoidal fuzzy numbers in the classical EOQ model. Chen et al. (1996) fuzzified demand, holding cost, ordering cost and backordering cost into trapezoidal fuzzy numbers in the classical EOQ model with backorder. Gen et al. (1997) considered an inventory control model where the input parameters are vague and are given by triangular fuzzy numbers, and presented a new method which uses interval mean value concept. Roy ve Maiti (1997) proposed fuzzy EOQ models and used fuzzy non-linear programming and fuzzy geometric programming to obtain the optimal order quantity in the fuzzy sense. Tang et al. (2000) developed an approach to model multi-product production planning problems with the objective of minimizing the total inventory cost with fuzzy demands and fuzzy capacities. I et al. (2002) considered the single-period inventory problem in the presence of uncertainties, two of which are randomness incorporated through the probability theory, and fuzziness characterized by fuzzy numbers. Chang (2004) extended the work by Salameh and Jaber (2000) considering the fuzziness in demand and the proportion of defective items. Two fuzzy inventory models were proposed. While the first model incorporates the fuzziness of the proportion of defective items, both the proportion of defective items and demand are represented as triangular fuzzy numbers in the second model. Dutta et al. (2005) presented a single-period inventory problem in the fuzzy environment by introducing demand as a fuzzy random variable. Mandal and Roy (2006) developed a multi-item inventory model under limited display-space constraint in fuzzy environment, where demand is considered as a function of the displayed inventory level and the cost parameters are assumed to be triangular fuzzy numbers. De and Goswami (2006) developed an EOQ model with fuzzy inflation rate and fuzzy deterioration rate under permissible delay in payment. Chen and Chang (2006)proposed a multi-product, multi-echelon, and multi-period supply chain model with fuzzy parameters, and developed a solution procedure that is able to calculate the fuzzy objective value of the fuzzy model.Das et al. (2007) proposed a single period production-inventory model with two warehouse, constant or stock dependent demand and imperfect production system under fuzzy budget constraint. Dutta et al. (2007) analyzed a single-period inventory model with a reordering strategy in a fuzzy environment, where demand is linguistic in nature and characterized as a fuzzy number. Wang et al. (2007) investigated an EOQ model for defective items, where the proportion of defective items in each lot is characterized as a random fuzzy variable while the cost parameters are characterized as fuzzy variables. They solved the fuzzy EOQ model by designing a particle swarm optimization (PSO) algorithm based on the random fuzzy simulation. Xu and Liu (2008) provided a method of solving solution sets of fuzzy multi-objective inventory problems with fuzzy random variables. Maity and Maiti (2008) considered a production-inventory system for deteriorating items with warehouse capacity and investment constraints in fuzzy environment. Liu (2008) developed a solution method to derive the fuzzy total profit of the classical inventory models when demand and cost parameters are fuzzy numbers. Vijayan and Kumaran (2008)developed fuzzy inventory models with partial backorders and lost sales where cost parameters are assumed to be trapezoidal fuzzy numbers. Björk (2009)proposed a fuzzy EOQ model in which demand and lead time are represented as triangular fuzzy numbers. Roy et al. (2009) investigated a production inventory model for a single product in an imperfect manufacturing system with remanufacturing of defective items considering the fuzziness in the proportion of defective items. Most recently, Karmakar et al. (2017) developed an EPQ model for deteriorating items in which shortages are allowed and partially backordered, production rate is a function of demand and cycle time and all cost parameters are fuzzy.

Besides, the fuzziness of decision variable(s) in inventory models is another research stream in the literature. Lin and Yao (2000) considered the classical EPQ problem in the fuzzy sense. They fuzzified production quantity into trapezoidal fuzzy number, and obtained the optimal production quantity using the extension principle and centroid method. Later, Hsieh (2002) developed two production inventory models for crisp or fuzzy production quantity. The fuzzy total cost functions are defuzzified using graded mean integration representation (GMIR) method, and the optimal solutions of these models are obtained by using Extension of the Lagrangean method for solving inequality constrain problem. Chen and Chang (2008) extended the classical EPQ model to the fuzzy environment considering crisp production quantity and fuzzy production quantity. Vijayan and Kumaran (2009) extended the work by Chen(2003)to the case where the cycle length, holding cost, setup cost and purchasing cost are fuzzy numbers, and used graded mean integration method to obtain the total cost in the fuzzy sense and the optimal cycle length. Kazemi et al. (2010) considered the classical EOQ model with backorders in fuzzy environment in which the optimal order quantity and the maximum backorder level are characterized as fuzzy variables. Recently, Glock et al. (2012) applied the concept of learning in fuzziness to the classical EOQ model with fuzzy demand. The total cost function was defuzzified by using graded mean integration representation, signed distance and centroid methods. Qin and Kar (2013) investigated the single-period inventory problem in the fuzziness of demand. Shekarian et al. (2014) extended the paper of Cárdenas-Barrón (2009) by fuzzifying input parameters into trapezoidal and triangular fuzzy numbers. Xu (2014) considered an integrated inventory problem under trade credit, where demand and the deterioration rate are characterized as fuzzy random variables with known probability distributions. The fuzzy random models based on three decision criteria which are expected value, chance-constrained and chance maximization were developed. Kazemi et al. (2015) extended the work by Salameh and Jaber (2000) to the fuzzy-learning environments. In a subsequent paper, Kazemi et al. (2016) discussed the effect of human learning on a fuzzy EOQ model with backorders.

To make the perfect decisions, the essential information may not be available or accessible to researchers or decision makers all the time. In inventory decisions, the policies could turn out to be imperfect by using the crisp values, and thus, upon the implementation of such imperfect policies, they may prove to be costly. Therefore, the input parameters' precision is improved by using the fuzzy set theory, which lessens the uncertainty and reduces errors.

In this paper, we revise the work by Jamal et al. (2004) and present a mathematical model to determine the optimal production time for a single-stage production system with rework process. We then consider the production inventory model in fuzzy environment. The fuzzy total inventory cost functions are derived for both inventory models with crisp and fuzzy production time, respectively, and the fuzzy total inventory cost functions are defuzzified in a similar manner to Hsieh (2002) and Vijayan and Kumaran (2009). In fuzzy production inventory model for crisp production time, the first derivative of fuzzy total production inventory cost is used to solve the optimal production time. Furthermore, the algorithm of Extension of the Lagrangean method is used to solve inequality constraints in fuzzy production inventory model for fuzzy production time.

The basic concept of the fuzzy set theory is the association of every object in the fuzzy set with an indicator value, which denotes the degree of its membership. In the written works, there exist different kinds of membership functions, which are classified as either a linear group or a non-linear group. Some of the linear membership functions make the calculations complex according to the studies, and therefore, the need of developing the simple fuzzy membership functions becomes imperative; in contrast, the selection of a non-linear membership function is not appropriate because it may not yield the perfect solution; however, few of them do result in a perfect solution to optimize such models, but they are costly in terms of time and effort to develop the complicated heuristic algorithms. Such approach is not only uneconomical but also impractical. Hence, these kind of member functions are not suitable for the model proposed in this paper. The fuzzy production inventory models are proposed in this paper, and their fuzzy production times and parameters are the fuzzy numbers that are types of linear membership functions; those numbers are as follows: triangular fuzzy numbers and trapezoidal fuzzy numbers.

The product of fuzzy numbers yields a fuzzy number with a very complex membership function if the extension principle is used as a fuzzy arithmetical operation. For instance, the product of one fuzzy number with a trapezoidal membership function with another fuzzy number with its trapezoidal membership function yields a fuzzy number with a membership function that has a shape like a parabolic drum with two sides. On the other hand, the product of one fuzzy number with a trapezoidal membership function with another fuzzy number with a membership function that has a shape like a parabolic drum with two sides yields a fuzzy number with a membership function that has a shape like a cube with two sides. When the original numbers get more complex, it becomes more complicated not only to operate on the fuzzy numbers but also to represent those numbers (Chen et al., 2006).

The Function Principle is proposed by Chen (1985) in order to handle the fuzzy arithmetical operations by using the trapezoidal fuzzy numbers. Instead of using the Extension Principle, Chen (1985) used the Function Principle in his paper. The first reason of using the Function Principle is that it is easier to use compared to the Extension Principle. Secondly, after the product of two trapezoidal fuzzy numbers, the Function Principle does not affect the shape of the trapezoidal fuzzy number. Thirdly, the multiplication of four trapezoidal fuzzy numbers or more can still be handled by using the Function Principle. As stated above, since the shape of triangular or trapezoidal fuzzy number is not affected by the Function Principle, it would be an appropriate method to use for the complicated model here to avoid the degenerate solutions. Moreover, there are various terms of multiplication operations of fuzzy numbers involved in the developed model proposed in this paper. For the aforementioned reasons, the Function Principle is preferred in order to avoid getting not only the very complicated mathematical expressions but also the degenerate solutions. Furthermore, by the virtue of the Function Principle, the fuzzy arithmetical operations are handled properly. Instead of using the Extension Principle, the Function Principle is used not only to make the computation of trapezoidal fuzzy numbers simpler but also to determine the fuzzy total cost of the production inventory of the developed model.

To make the usage of the fuzzy set theory easy for the decision makers, the fuzzy outcomes should be transformed into crisp values. The extraction process of the crisp values from the fuzzy models is called defuzzification. The famous defuzzification methods or processes are as follows: signed distance method, centroid method, and the GMIR method. The efficient and productive defuzzification process is suggested to be selected so as to defuzzify the fuzzy inventory function. The GMIR method, which was developed by Chen and Hsieh (1998), is applied here in order to make the mathematical process simpler and to improve the application of the model. The fuzzy cost function is defuzzified by using the GMIR method. Using this method, the defuzzified value can be evaluated by GMIR directly by utilizing the fuzzy arithmetic operations as the membership function is not affected by them.

The remainder of the paper is organized as follows. In Section 2, we present the assumptions and notations for the production inventory model, and derive the total production inventory cost function. In addition, we prove the convexity of the total cost per unit of time function, and obtain the optimal production time period which is unique. In Section 3, some basic concepts of fuzzy sets theory and the Lagrangean optimization method are presented. In Section 4, the fuzzy total cost functions are derived and the graded mean integration representation method is applied to defuzzify the functions. In Section 5, numerical examples are provided to discuss the effects of the fuzziness of the components over the optimal production time period and the total inventory cost. The last section concludes and summarizes the paper.

2. Development of the production inventory model

The EOQ and EPQ models are developed in order to facilitate the researchers and the decision makers to comprehend the working of the production inventory systems and the cost of those systems. Despite the simplistic nature of these models and their functionality, they incorporate the limited conditions. They do not include the defective or imperfect products in a lot size. Few factors are uneconomical and can impact the decisions or production. Those factors may include the following: untrained or unqualified operators of the production systems, faulty machines, below-standard raw materials, and defective process that lead to producing imperfect or defective products. Different costs are curtailed by doing the rework process on the defective items where there is a need of a trade-off. Likewise, for the imperfect production process, Jamal et al. (2004) has developed an EPQ inventory model, in which the rework process of the defective items is done to renew and add them in the inventory for the demand of the buyers.

We use the same notation and assumptions as in Jamal et al. (2004).Jamal et al. (2004) use the production quantity Q as decision variable. However, in this study, we use the production time t, i.e. production uptime, as decision variable.

Notations*:*

P The production rate in units per unit time
Q Production lot size per cycle (dependent variable)
D The demand rate in units per unit time
t The production time (decision variable)
t_1 The time period needed to rework of defective items
t_2 The time period when inventory depletes
T Cycle length, $T = t + t_1 + t_2$
β The proportion of defective items produced
S The setup cost for each production
Q_1 The maximum level of on-hand inventory of perfect items in units, when the regular production process stops,
Q_2 The maximum level of on-hand inventory of perfect items in units, when the rework process ends,
C The processing cost per unit (\$/unit)
H The holding cost per unit per unit time (\$/unit/unit time)
$*$ The superscript representing optimal value

Assumptions*:*

- Demand is constant and continuous.
- Production rate is constant and is greater than the demand rate, $P > D$.
- Production and rework are done at the same speed.
- Each lot produced contains defective items with a proportion ofβ.
- 100% recovery is provided through the reworking.
- Inspection cost is ignored.
- Shortages are not allowed. To avoid shortages during the regular production process, the production rate must always greater than the demand rate. That is,

$$P(1-\beta) > D. \tag{2.1}$$

Figure 1 depicts the production inventory model for defective items reworked within the same cycle. During the regular production process t, good items are produced with a unit production cost$C$. It is assumed that production process generates defective items with a proportion ofβwhich is constant. When the regular production process ends the rework process is accomplished and all defective items are made as-good-as-perfect through the reworking during the time periodt_1. The consumption of on-hand inventory continues at the end of time periodt_2.

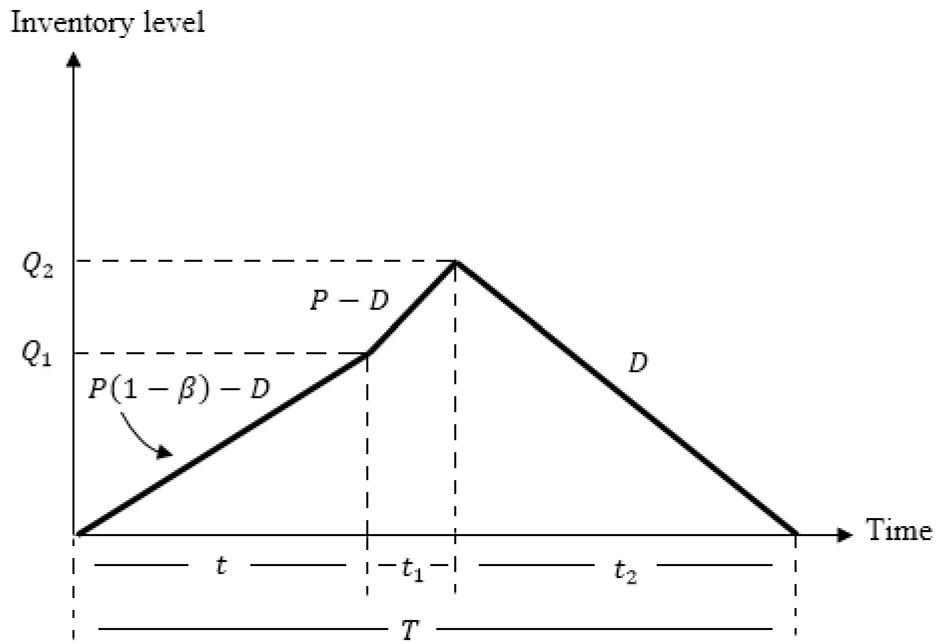

Figure1: The behavior of the inventory level for the single-stage production inventory model with rework process.

Referring to the Figure 1, the following equations which are production uptime$t$, the time period $t_1$required to rework defective items produced during the production time t, production downtime t_2 and the inventory levels of Q_1 and Q_2 can be derived.

$$t = \frac{Q}{P}. \tag{2.2}$$

$$Q = tP \tag{2.3}$$

$$t_1 = \frac{Q_2 - Q_1}{P - D} \tag{2.4}$$

$$t_2 = \frac{Q_2}{PD} = T - t - t_1 \tag{2.5}$$

$$Q_1 = (P(1-\beta) - D)t, \tag{2.6}$$

$$Q_2 = Q_1 + (P - D)t_1, \tag{2.7}$$

The time period $t_1$needed to reworkβQ units of items is computed as in Equation (2.8):

$$t_1 = \frac{\beta Q}{P} = \beta t, \tag{2.8}$$

Thus, according to Equations (2.2) and (2.8), the inventory level Q_2 is obtained as follows:

$$Q_2 = (P(1-\beta) - D)t + (P - D)\beta t. \tag{2.9}$$

The total inventory cost per cycle includes the production setup cost, the production cost, the reworking cost and the holding cost. Let$TC(t)$denote the total invetory cost per cycle. Then,

$$\begin{aligned} TC(t) &= S + CPt + C\beta Pt + H\left(\frac{Q_1(t)}{2} + \frac{(Q_1 + Q_2)(t_1)}{2} + \frac{Q_2(t_2)}{2}\right) \\ &= S + (1+\beta)CPt + \frac{HPt^2}{2D}\big((1+\beta)(P(1-\beta) - D) + \beta^2(P - D)\big) \end{aligned} \tag{2.10}$$

The total cycle length is the summation of the production uptime, the reworking time and the production downtime:

$$T = t + t_1 + t_2 = \frac{Q}{D} = \frac{tP}{D}. \tag{2.11}$$

Then,using the renewal reward theorem, the total inventory cost per unit time is given as

$$\begin{aligned} TCU(t) &= \frac{TC(t)}{T} \\ &= \frac{SD}{tP} + CD(1+\beta) + \frac{Ht}{2}\big((1+\beta)(P(1-\beta) - D) + \beta^2(P - D)\big). \end{aligned} \tag{2.12}$$

It can be shown that the total cost per unit time function $TCU(t)$ is convex with respect to t. Hence, the optimal production time t_c^* can be found by taking the first derivative of the $TCU(t)$ with respect to t and equate the result to zero, i.e.

$$\frac{\partial TCU(t)}{\partial t} = -\frac{SD}{t^2 P} + \frac{H}{2}\big((1+\beta)(P(1-\beta) - D) + \beta^2(P - D)\big) = 0 \tag{2.13}$$

Solving the Equation (2.13) for t, the optimal production time is

$$t_c^* = \sqrt{\frac{2SD}{HP\big((1+\beta)(P(1-\beta) - D) + \beta^2(P - D)\big)}}. \tag{2.14}$$

It is important to mention that the above model can be considered as an EPQ model given by Jamal et al. (2004)when$t$is replaced byQ/P. Further, assume that the proportion of defective items produced in regular production process is zero, i.e. $\beta = 0$. Hence, Equation (2.14) reduces to the classical economic production quantity

$$Q^* = \sqrt{\frac{2SD}{H\left(1 - \frac{D}{P}\right)}}. \tag{2.15}$$

In Section 4, the fuzzy equivalent of the model given above by Equation (2.12) will be discussed. The fuzzy set theory is introduced in the next section.

3. Preliminaries concepts

In this section we give the definition of fuzzy numbers and fuzzy arithmetical operations in order to make the fuzzy inventory models proposed in this paper.

3.1. Definition A fuzzy set $\tilde{a}_\alpha$ on R is called a $\alpha-$*cut fuzzy point*, where $\alpha \in [0,1]$, provided that the membership function of $\tilde{a}_\alpha$ is

$$\mu_{\tilde{a}_\alpha}(x) = \begin{cases} \alpha, & x = a, \\ 0, & x \neq a. \end{cases} \tag{3.1}$$

3.2. Definition A fuzzy set $\tilde{E} = (a, b, c, d)$ on R, where, $a < b < c < d$ is called a *trapezoidal fuzzy number* if its membership function is

$$\mu_{\tilde{E}}(x) = \begin{cases} p(x) = \dfrac{x-a}{b-a} & a \leq x \leq b, \\ 1 & b \leq x \leq c, \\ r(x) = \dfrac{d-x}{d-c} & c \leq x \leq d, \\ 0, & otherwise. \end{cases} \tag{3.2}$$

3.3. Definition A fuzzy interval $[a_\alpha, b_\alpha]$ on R is called $\alpha-$*level fuzzy interval* if its membership function is

$$\mu_{[a_\alpha, b_\alpha]}(x) = \begin{cases} \alpha, & a \leq x \leq b \\ 0, & other\ wise \end{cases} \tag{3.3}$$

The $\alpha-$cut of a fuzzy number $\tilde{E}$ is defined as

$$E(\alpha) = \{x | \mu_{\tilde{E}}(x) \geq \alpha\} = [E_L(\alpha), E_U(\alpha)] \tag{3.4}$$

Denote F as the family of fuzzy numbers on R. For each fuzzy set $\tilde{E} \in F$, the set $E(\alpha)$ is a unique closed interval. The membership function of the fuzzy number $\tilde{E}$ is

$$\begin{aligned} \mu_{\tilde{E}}(x) &= \bigcup_{\alpha \in [0,1]} \alpha \wedge C_{E_\alpha}(x) = \bigcup_{\alpha \in [0,1]} \mu_{[E_L(\alpha), E_U(\alpha)]}(x) \\ &= \mu_{\bigcup_{\alpha \in [0,1]} [E_L(\alpha), E_U(\alpha)]}(x) \end{aligned} \tag{3.5}$$

and a fuzzy set $\tilde{E}$ is described as

$$\tilde{E} = \bigcup_{\alpha \in [0,1]} [E_L(\alpha), E_U(\alpha)] \tag{3.6}$$

Where $C_{E(\alpha)}(x) = \begin{cases} 1, & x \in E_\alpha \\ 0, & x \notin E_\alpha \end{cases}$.

Now, we define an algebraic operations related with closed interval of real numbers. Any point a, b, c, d and k on R and $a < b$ and $c < d$, the operations of addition, subtraction and scalar multiplication defined as

$$[a,b]+[c,d]=[a+b,c+d],$$
$$[a,b]-[c,d]=[a-d,b-c], \tag{3.7}$$
$$k*[a,b]=\begin{cases}[ka,kb], k>0,\\ [kb,ka], \quad k<0.\end{cases}$$

Multiplication and division operations are also described as

$$[a,b]*[c,d]=[min\{ac,ad,bc,bd\},max\{ac,ad,bc,bd\}],$$
$$\frac{[a,b]}{[c,d]}=\left[min\left\{\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right\},max\left\{\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right\}\right]. \tag{3.8}$$

In particular, let$a>0$, $c>0$and$c,d\neq 0$, then the following equations hold

$$[a,b]*[c,d]=[ac,bd],$$
$$\frac{[a,b]}{[c,d]}=\left[\frac{a}{d},\frac{b}{c}\right]. \tag{3.9}$$

3.4. The fuzzy arithmetical operations under function principle

Chen (1985) has introduced the Function principle as the fuzzy arithmetical operations by using the trapezoidal fuzzy numbers. In this paper, Function Principle's arithmetical operations will be used. Function Principle's arithmetical operations are elaborated as follows:

Assume that $\tilde{E}=(e_1,e_2,e_3,e_4)$ and $\tilde{F}=(f_1,f_2,f_3,f_4)$ are two trapezoidal fuzzy numbers. Then, we have

- The addition of $\tilde{E}$ and $\tilde{F}$ is

  $\tilde{E}\oplus\tilde{F}=(e_1+f_1,e_2+f_2,e_3+f_3,e_4+f_4),$

where $e_1,e_2,e_3,e_4,f_1,f_2,f_3$ and f_4 are any real numbers.

- If $e_i,f_i>0$, for $i=1,2,3,4$, then the multiplication of $\tilde{E}$ and $\tilde{F}$ is
  $\tilde{E}\otimes\tilde{F}=(e_1f_1,e_2f_2,e_3f_3,e_4f_4).$

- Since $\ominus\tilde{F}=(-f_4,-f_3,-f_2,-f_1)$, the subtraction of $\tilde{E}$ and $\tilde{F}$ is
  $\tilde{E}\ominus\tilde{F}=\tilde{E}\oplus\left(\ominus\tilde{F}\right)=(e_1-f_4,e_2-f_3,e_3-f_2,e_4-f_1).$

- $1\oslash\tilde{F}=\tilde{F}^{-1}=\left(\frac{1}{f_4},\frac{1}{f_3},\frac{1}{f_2},\frac{1}{f_1}\right),$
  where$e_i>0$ for each $i=1,2,3,4$.

  If $e_i,f_i>0$ for $i=1,2,3,4$, then we get
  $\tilde{E}\oslash\tilde{F}=\left(\frac{e_1}{f_4},\frac{e_2}{f_3},\frac{e_3}{f_2},\frac{e_4}{f_1}\right).$

- Let $\ell \in R$, then

$$\ell \odot \tilde{E} = \ell \otimes \tilde{E} = \begin{cases} (\ell e_1, \ell e_2, \ell e_3, \ell e_4) & \ell \geq 0, \\ (\ell e_4, \ell e_3, \ell e_2, \ell e_1) & \ell < 0. \end{cases}$$

3.5. Graded Mean Integration Representation (GMIR) Method

In order to defuzzify the fuzzy total cost function, as in Hsieh (2002) and Vijayan and Kumaran (2009), we will use the graded mean integration representation method introduced by Chen and Hsieh (1998).

For the fuzzy number $\tilde{E}$ in Equation (3.2), let the functions p^{-1} and r^{-1} are the inverse functions of p and r (left and right function of $\tilde{E}$), respectively. We also describe the graded mean α −level value of $\tilde{E}$ as

$$\frac{\alpha((p^{-1}(\alpha) + r^{-1}(\alpha))}{2} \tag{3.10}$$

For the trapezoidal fuzzy number $\tilde{E} = (a, b, c, d)$, the graded mean integration representation of $\tilde{E}$ can be obtained as

$$DF(\tilde{E}) = \frac{\int_0^1 \frac{\alpha((p^{-1}(\alpha)+r^{-1}(\alpha))}{2} d\alpha}{\int_0^1 \alpha d\alpha} = \int_0^1 \alpha((p^{-1}(\alpha) + r^{-1}(\alpha))\, d\alpha$$

$$= \frac{a + 2b + 2c + d}{6}, \tag{3.11}$$

where $p^{-1}(\alpha) = a - (a - b)\alpha, r^{-1}(\alpha) = d + (c - d)\alpha$.

Now assume $b = c = e$, then the graded mean integration representation of triangular fuzzy number $\tilde{E}_1 = (a, e, d)$,

$$DF(\tilde{E}_1) = \frac{a + 4e + d}{6}. \tag{3.12}$$

3.6. Lagrangean Optimization

The Karush-Kuhn-Tucker (KKT) conditions are an extension of the Lagrangean method to the nonlinear programming problems with inequality constraints, and discussed in Taha (2011). For the minimization problem,

$$Min\ y = f(x)$$
$$\text{Subject to } g_i(x) \geq 0,\ i = 1,2, \dots, m.$$
$$x \geq 0,$$

The procedure of extension of Lagrangean method involves the following steps:

Step 1. Solve the unconstrained problem $Min\ y = f(x)$. If the resulting solution satisfies all the constraints, stop the procedure, the solution is optimum. Otherwise set the number of constraints $K = 1$ and go to Step 2.

Step 2. Activate any K constraints by converting them into equalities and minimize $f(x)$ subject to the K active constraints by the Lagrangean method. If the resulting solution is feasible with respect to the remaining constraints, stop the procedure, it is a local optimum. Otherwise, take another set of K constraints and repeat the step. If all sets of active constraints taken at a time are considered without encountering a feasible solution, go to Step 3.

Step 3. If $K = m$, stop the procedure, no feasible solution exists. Otherwise set $K = K + 1$ and go to Step 2.

4. Fuzzy production inventory models

In order to find the probability distribution of the input parameters, such as the defective rate etc., the value changes of those parameters would be not sufficiently enough. Furthermore, in reality, uncertainty cannot be avoided due to the imprecision and vagueness of the parameter(s) or the decision variable(s) of the inventory model. Considering the real-life scenarios, the stochastic methods as well as the statistical methods may turn out to be non-productive solutions for the development of the inventory models based on the reality. The efficient method to express the various factors, which are neither, be evaluated by the crisp values nor by the random processes, like the products' quality, stock out situation in inventory, enhanced demand response, and flexibility, is one of the following: linguistic variables, or the fuzzy numbers.

Björk et al. (2012) mentioned that the uncertainties originate from different sources, but in many situations, the demand as well as the cycle time is fuzzy. In the context of Nordic process industry, the uncertainties in cycle time originates from different sources; for example, sometimes because of the unavailability of raw material that is essential at the production time, and, hence, the production planning increases the number of items produced on a particular machine at the given time. Such kind of reasoning originates from the fact that the production units, which are capital intensive, may turn out to be uneconomical because of their idleness. Furthermore, the setup times of replacing products may also be significant. Few other factors also contribute to the overall situation that the complete cycle time and the demand are often fuzzy by default. Hence, this paper assumes that the production time is represented as a fuzzy number.

The following two cases regarding the defective productions, which can be reworked or repaired, will be discussed in this paper: the first case considers the fuzzy parameters and the crisp production time, whereas the second case takes the fuzzy parameters with the fuzzy production time.

4.1. Fuzzy production inventory model for crisp production time

In this subsection, we analyze the deterministic inventory model given in Equation (2.12) with combining the fuzziness of all the input parameters (S, D, P, C, β, H). In this case, the total cost per unit of time in the fuzzy sense is given as

$$\tilde{G}(t) = \frac{\tilde{S} * \tilde{D}}{t * \tilde{P}} + \tilde{C} * \tilde{D} * \left(1 + \tilde{\beta}\right) + \frac{\tilde{H} * t}{2}\left\{\left(1 + \tilde{\beta}\right) * \left(\left(1 - \tilde{\beta}\right) * \tilde{P} - \tilde{D}\right) + \tilde{\beta}^2 * \left(\tilde{P} - \tilde{D}\right)\right\} \tag{4.1.1}$$

where $+,\ -,\ *$ and $/$ are the fuzzy arithmetical operations. In this equation, we assume that $\tilde{S} = (S - S_1, S - S_2, S + S_3, S + S_4)$, $\tilde{D} = (D - D_1, D - D_2, D + D_3, D + D_4)$, $\tilde{P} = (P - P_1, P - P_2, P + P_3, P + P_4)$, $\tilde{C} = (C - C_1, C - C_2, C + C_3, C + C_4)$,

$\tilde{\beta} = (\beta - \beta_1, \beta - \beta_2, \beta + \beta_3, \beta + \beta_4), \widetilde{H} = (H - v_1, H - v_2, H + v_3, H + v_4)$ are trapezoidal fuzzy numbers. For $i = 1,2,3,4, S_i, D_i, P_i, C_i, \beta_i$ and v_i are arbitrary positive numbers which satisfy $S_1 > S_2$, $S_3 < S_4$, $D_1 > D_2$, $D_3 < D_4$, $P_1 > P_2$, $P_3 < P_4$, $C_1 > C_2$, $C_3 < C_4$, $\beta_1 > \beta_2$, $\beta_3 < \beta_4$, $v_1 > v_2$ and $v_3 < v_4$.

The following equations can be written using fuzzy arithmetical operations as the trapezoidal fuzzy numbers

$$\tilde{S} * \widetilde{D} = \big((S - S_1)(D - D_1), (S - S_2)(D - D_2), (S + S_3)(D + D_3), (S + S_4)(D + D_4)\big) \tag{4.1.2}$$

$$t * \tilde{P} = \big(t(P - P_1), t(P - P_2), t(P + P_3), t(P + P_4)\big) \tag{4.1.3}$$

$$\frac{\tilde{S} * \widetilde{D}}{t * \tilde{P}} = \left(\frac{(S - S_1)(D - D_1)}{t(P + P_4)}, \frac{(S - S_2)(D - D_2)}{t(P + P_3)}, \frac{(S + S_3)(D + D_3)}{t(P - P_2)}, \frac{(S + S_4)(D + D_4)}{t(P - P_1)}\right) \tag{4.1.4}$$

$$1 + \tilde{\beta} = (1 + \beta - \beta_1, 1 + \beta - \beta_2, 1 + \beta + \beta_3, 1 + \beta + \beta_4) \tag{4.1.5}$$

$$\tilde{C} * \widetilde{D} * (1 + \tilde{\beta}) = \big((C - C_1)(D - D_1)(1 + \beta - \beta_1), (C - C_2)(D - D_2)(1 + \beta - \beta_2), (C + C_3)(D + D_3)(1 + \beta + \beta_3), (C + C_4)(D + D_4)(1 + \beta + \beta_4)\big), \tag{4.1.6}$$

$$\frac{\widetilde{H} * t}{2} = \left(\frac{(H - v_1)t}{2}, \frac{(H - v_2)t}{2}, \frac{(H + v_3)t}{2}, \frac{(H + v_4)t}{2}\right), \tag{4.1.7}$$

$$1 - \tilde{\beta} = \big(1 - (\beta + \beta_4), 1 - (\beta + \beta_3), 1 - (\beta - \beta_2), 1 - (\beta - \beta_1)\big) \tag{4.1.8}$$

$$(1 - \tilde{\beta}) * \tilde{P} - \widetilde{D} = \Big(\big(1 - (\beta + \beta_4)\big)(P - P_1) - (D + D_4), \big(1 - (\beta + \beta_3)\big)(P - P_2) - (D + D_3), 1 - (\beta - \beta_2)(P + P_3) - (D - D_2), \big(1 - (\beta - \beta_1)\big)(P + P_4) - (D - D_1)\Big), \tag{4.1.9}$$

$$(1 + \tilde{\beta}) * \big((1 - \tilde{\beta}) * \tilde{P} - \widetilde{D}\big)$$

$$= \Big((1+\beta-\beta_1)\Big(\big(1-(\beta+\beta_4)\big)(P-P_1)-(D+D_4)\Big), (1+\beta - \beta_2)\Big(\big(1-(\beta+\beta_3)\big)(P-P_2)-(D+D_3)\Big), (1+\beta + \beta_3)\Big(\big(1-(\beta-\beta_2)\big)(P+P_3)-(D-D_2)\Big), (1+\beta + \beta_4)\Big(\big(1-(\beta-\beta_1)\big)(P+P_4)-(D-D_1)\Big)\Big) \tag{4.1.10}$$

$$\tilde{\beta}^2 = ((\beta-\beta_1)^2, (\beta-\beta_2)^2, (\beta+\beta_3)^2, (\beta+\beta_4)^2) \tag{4.1.11}$$

$$\tilde{\beta}^2 * \big(\tilde{P}-\tilde{D}\big) = \Big((\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big), (\beta-\beta_2)^2\big(P-P_2 - (D+D_3)\big), (\beta+\beta_3)^2\big(P+P_3-(D-D_2)\big), (\beta+\beta_4)^2\big(P+P_4 - (D-D_1)\big)\Big). \tag{4.1.12}$$

Substituting Equations (4.1.2)-(4.1.12) in Equation (4.1.1), we have the trapezoidal fuzzy number

$$\tilde{\mathcal{G}}(t) = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4) \tag{4.1.13}$$

where

$$\mathcal{G}_1 = \frac{(S-S_1)(D-D_1)}{t(P+P_4)} + (C-C_1)(D-D_1)(1+\beta-\beta_1) + \frac{(H-v_1)t}{2}\Big((1+\beta-\beta_1)\Big(\big(1-(\beta+\beta_4)\big)(P-P_1)-(D+D_4)\Big) + (\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big),$$

$$\mathcal{G}_2 = \frac{(S-S_2)(D-D_2)}{t(P+P_3)} + (C-C_2)(D-D_2)(1+\beta-\beta_2) + \frac{(H-v_2)t}{2}\Big((1+\beta-\beta_2)\Big(\big(1-(\beta+\beta_3)\big)(P-P_2)-(D+D_3)\Big) + (\beta-\beta_2)^2\big(P-P_2-(D+D_3)\big)\Big),$$

$$\mathcal{G}_3 = \frac{(S+S_3)(D+D_3)}{t(P-P_2)} + (C+C_3)(D+D_3)(1+\beta+\beta_3) + \frac{(H+v_3)t}{2}\Big((1+\beta+\beta_3)\Big(\big(1-(\beta-\beta_2)\big)(P+P_3)-(D-D_2)\Big) + (\beta+\beta_3)^2\big(P+P_3-(D-D_2)\big)\Big),$$

$$\mathcal{G}_4 = \frac{(S+S_4)(D+D_4)}{t(P-P_1)} + (C+C_4)(D+D_4)(1+\beta+\beta_4) + \frac{(H+v_4)t}{2}\Big((1+\beta+\beta_4)\Big(\big(1-(\beta-\beta_1)\big)(P+P_4)-(D-D_1)\Big) + (\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big).$$

The graded mean integration representation of the fuzzy number $\tilde{G}(t)$ is obtained from Equation (3.11) as

$$\begin{aligned}
DF\left(\tilde{\mathcal{G}}(t)\right) = {} & \frac{1}{6}\Bigg[\frac{(S-S_1)(D-D_1)}{t(P+P_4)} + (C-C_1)(D-D_1)(1+\beta-\beta_1) \\
& + \frac{(H-v_1)t}{2}\Big((1+\beta-\beta_1)\Big(\big(1-(\beta+\beta_4)\big)(P-P_1) - (D+D_4)\Big) + (\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big)\Bigg] \\
& + \frac{2}{6}\Bigg[\frac{(S-S_2)(D-D_2)}{t(P+P_3)} + (C-C_2)(D-D_2)(1+\beta-\beta_2) \\
& + \frac{(H-v_2)t}{2}\Big((1+\beta-\beta_2)\Big(\big(1-(\beta+\beta_3)\big)(P-P_2) - (D+D_3)\Big) + (\beta-\beta_2)^2\big(P-P_2-(D+D_3)\big)\Big)\Bigg] \\
& + \frac{2}{6}\Bigg[\frac{(S+S_3)(D+D_3)}{t(P-P_2)} + (C+C_3)(D+D_3)(1+\beta+\beta_3) \\
& + \frac{(H+v_3)t}{2}\Big((1+\beta+\beta_3)\Big(\big(1-(\beta-\beta_2)\big)(P+P_3) - (D-D_2)\Big) + (\beta+\beta_3)^2\big(P+P_3-(D-D_2)\big)\Big)\Bigg] \\
& + \frac{1}{6}\Bigg[\frac{(S+S_4)(D+D_4)}{t(P-P_1)} + (C+C_4)(D+D_4)(1+\beta+\beta_4) \\
& + \frac{(H+v_4)t}{2}\Big((1+\beta+\beta_4)\Big(\big(1-(\beta-\beta_1)\big)(P+P_4) - (D-D_1)\Big) + (\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big)\Bigg].
\end{aligned} \quad (4.1.14)$$

Our objective is to minimize the defuzzified total cost function$DF\left(\tilde{\mathcal{G}}(t)\right)$. In order to show that$DF\left(\tilde{\mathcal{G}}(t)\right)$is convex, the first and second order partial derivatives of $DF\left(\tilde{\mathcal{G}}(t)\right)$ with respect to$t$ are given the following equations, respectively.

$$\begin{aligned}\frac{\partial DF\left(\tilde{\mathcal{G}}(t)\right)}{\partial t} &= \frac{1}{6}\Bigg[-\frac{(S-S_1)(D-D_1)}{t^2(P+P_4)} \\ &\quad +\frac{(H-v_1)}{2}\Big((1+\beta-\beta_1)\big((1-(\beta+\beta_4))(P-P_1)-(D+D_4)\big) \\ &\quad +(\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big)\Bigg] \\ &\quad +\frac{2}{6}\Bigg[-\frac{(S-S_2)(D-D_2)}{t^2(P+P_3)} \\ &\qquad +\frac{(H-v_2)}{2}\Big((1+\beta-\beta_2)\big((1-(\beta+\beta_3))(P-P_2) \\ &\qquad -(D+D_3)\big)+(\beta-\beta_2)^2\big(P-P_2-(D+D_3)\big)\Big)\Bigg] \\ &\quad +\frac{2}{6}\Bigg[-\frac{(S+S_3)(D+D_3)}{t^2(P-P_2)} \\ &\qquad +\frac{(H+v_3)}{2}\Big((1+\beta+\beta_3)\big((1-(\beta-\beta_2))(P+P_3) \\ &\qquad -(D-D_2)\big)+(\beta+\beta_3)^2\big(P+P_3-(D-D_2)\big)\Big)\Bigg] \\ &\quad +\frac{1}{6}\Bigg[-\frac{(S+S_4)(D+D_4)}{t^2(P-P_1)} \\ &\qquad +\frac{(H+v_4)}{2}\Big((1+\beta+\beta_4)\big((1-(\beta-\beta_1))(P+P_4) \\ &\qquad -(D-D_1)\big)+(\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big)\Bigg],\end{aligned}$$

After some development, we have

$$\begin{aligned}\frac{\partial DF\left(\tilde{\mathcal{G}}(t)\right)}{\partial t} &= \frac{1}{12}\Big[(H-v_1)\Big((1+\beta-\beta_1)\big((1-(\beta+\beta_4))(P-P_1)-(D+D_4)\big) \\ &\quad +(\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big) \\ &\quad +2(H-v_2)\Big((1+\beta-\beta_2)\big((1-(\beta+\beta_3))(P-P_2)-(D+D_3)\big) \\ &\qquad +(\beta-\beta_2)^2\big(P-P_2-(D+D_3)\big)\Big)\end{aligned}$$

$$+2(H+v_3)\Big((1+\beta+\beta_3)\big((1-(\beta-\beta_2))(P+P_3)-(D-D_2)\big) + (\beta+\beta_3)^2\big(P+P_3-(D-D_2)\big)\Big)$$

$$+(H+v_4)\Big((1+\beta+\beta_4)\big((1-(\beta-\beta_1))(P+P_4)-(D-D_1)\big) + (\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big)\Big]$$

$$-\frac{1}{6t^2}\left(\frac{(S-S_1)(D-D_1)}{(P+P_4)}+\frac{2(S-S_2)(D-D_2)}{(P+P_3)}+\frac{2(S+S_3)(D+D_3)}{(P-P_2)} + \frac{(S+S_4)(D+D_4)}{(P-P_1)}\right)=0 \tag{4.1.15}$$

and the second-order partial derivative of $DF\left(\tilde{\mathcal{G}}(t)\right)$ is

$$\frac{\partial^2 DF\left(\tilde{\mathcal{G}}(t)\right)}{\partial t^2}=\frac{1}{3t^3}\left(\frac{(S-S_1)(D-D_1)}{(P+P_4)}+\frac{2(S-S_2)(D-D_2)}{(P+P_3)} + \frac{2(S+S_3)(D+D_3)}{(P-P_2)}+\frac{(S+S_4)(D+D_4)}{(P-P_1)}\right). \tag{4.1.16}$$

Note that, the second order partial derivative $\partial^2 DF\left(\tilde{\mathcal{G}}(t)\right)/\partial t^2$ is continuous and positive for all $t>0$.This implies that the function in Equation (4.1.16) is strictly convex for positive t. The optimal production time $t^*$is found by solving the first order partial derivative of Equation (4.1.16) equal to zero. This yield to

$$t^*=\sqrt{\frac{2\left(\frac{(S-S_1)(D-D_1)}{(P+P_4)}+\frac{2(S-S_2)(D-D_2)}{(P+P_3)}+\frac{2(S+S_3)(D+D_3)}{(P-P_2)}+\frac{(S+S_4)(D+D_4)}{(P-P_1)}\right)}{F_1+2F_2+2F_3+F_4}}, \tag{4.1.17}$$

where

$$F_1=(H-v_1)\Big((1+\beta-\beta_1)\big((1-(\beta+\beta_4))(P-P_1)-(D+D_4)\big) + (\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big),$$

$$F_2=(H-v_2)\Big((1+\beta-\beta_2)\big((1-(\beta+\beta_3))(P-P_2)-(D+D_3)\big) + (\beta-\beta_2)^2\big(P-P_2-(D+D_3)\big)\Big),$$

$$F_3=(H+v_3)\Big((1+\beta+\beta_3)\big((1-(\beta-\beta_2))(P+P_3)-(D-D_2)\big) + (\beta+\beta_3)^2\big(P+P_3-(D-D_2)\big)\Big).$$

$$F_4 = (H+v_4)\Big((1+\beta+\beta_4)\big((1-(\beta-\beta_1))(P+P_4)-(D-D_1)\big) + (\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big).$$

The value of defuzzified total cost function $DF\left(\tilde{\mathcal{G}}(t)\right)$ is obtained by direct substitution of Equation (4.1.17) into Equation (4.1.14).

When the input parameters (S, D, P, C, β, H) are real numbers, that is
$S = S - S_1 = S - S_2 = S + S_3 = S + S_4$, $D = D - D_1 = D - D_2 = D + D_3 = D + D_4$,
$P = P - P_1 = P - P_2 = P + P_3 = P + P_4$, $C = C - C_1 = C - C_2 = C + C_3 = C + C_4$
$\beta = \beta - \beta_1 = \beta - \beta_2 = \beta + \beta_3 = \beta + \beta_4$, $H = H - v_1 = H - v_2 = H + v_3 = H + v_4$,

Then the deterministic production inventory model is presented. So, the following reduced form of Equation (4.1.17) is obtained:

$$t_c^* = \sqrt{\frac{2SD}{PH\left((1+\beta)\big((1-\beta)P-D\big)+\beta^2(P-D)\right)}}. \tag{4.1.18}$$

If we assume that the input parameters are triangular fuzzy numbers as
$\tilde{S} = (S - S_1, S, S + S_4)$, $\tilde{D} = (D - D_1, D, D + D_4)$, $\tilde{P} = (P - P_1, P, P + P_4)$, $\tilde{C} = (C - C_1, C, C + C_4)$, $\tilde{\beta} = (\beta - \beta_1, \beta, \beta + \beta_4)$ and $\tilde{H} = (H - v_1, H, H + v_4)$,
then the following equations are attained

$$\tilde{S} * \tilde{D} = \big((S-S_1)(D-D_1), SD, (S+S_4)(D+D_4)\big) \tag{4.1.19}$$

$$t * \tilde{P} = \big(t(P-P_1), tP, t(P+P_4)\big) \tag{4.1.20}$$

$$\frac{\tilde{S} * \tilde{D}}{t * \tilde{P}} = \left(\frac{(S-S_1)(D-D_1)}{t(P+P_4)}, \frac{SD}{tP}, \frac{(S+S_4)(D+D_4)}{t(P-P_1)}\right), \tag{4.1.21}$$

$$1 + \tilde{\beta} = (1+\beta-\beta_1, 1+\beta, 1+\beta+\beta_4) \tag{4.1.22}$$

$$\tilde{C} * \tilde{D} * (1+\tilde{\beta}) = \Big(\big((C-C_1)(D-D_1)(1+\beta-\beta_1), CD(1+\beta), (C+C_4)(D+D_4)(1+\beta+\beta_4)\big)\Big), \tag{4.1.23}$$

$$\frac{\tilde{H} * t}{2} = \left(\frac{(H-v_1)t}{2}, \frac{Ht}{2}, \frac{(H+v_4)t}{2}\right), \tag{4.1.24}$$

$$1 - \tilde{\beta} = \big(1-(\beta+\beta_4), 1-\beta, 1-(\beta-\beta_1)\big) \tag{4.1.25}$$

$$\big(1-\tilde{\beta}\big) * \tilde{P} - \tilde{D} \tag{4.1.26}$$

$$= \Big(\big(1-(\beta+\beta_4)\big)(P-P_1)-(D+D_4), (1-\beta)P-D,$$

$$\big(1-(\beta-\beta_1)\big)(P+P_4)-(D-D_1)\Big),$$

$(1+\tilde{\beta})*\left((1-\tilde{\beta})*\tilde{P}-\tilde{D}\right)$

$$= \Big((1+\beta-\beta_1)\left(\big(1-(\beta+\beta_4)\big)(P-P_1)-(D+D_4)\right), (1+\beta)\big((1-\beta)P-D\big), (1+\beta+\beta_4)\left(\big(1-(\beta-\beta_1)\big)(P+P_4)-(D-D_1)\right)\Big), \quad (4.1.27)$$

$$\tilde{\beta}^2 = ((\beta-\beta_1)^2, \beta^2, \quad (\beta+\beta_4)^2) \quad (4.1.28)$$

$$\tilde{\beta}^2*\left(\tilde{P}-\tilde{D}\right) = \Big((\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big), \beta^2(P-D), (\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big). \quad (4.1.29)$$

Substituting Equations (4.1.19)-(4.1.29) in Equation (4.1.1), then the fuzzy total inventory cost is represented by a triangular fuzzy number as

$$\tilde{\mathcal{G}}(t) = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) \quad (4.1.30)$$

Where

$$\mathcal{G}_1 = \frac{(S-S_1)(D-D_1)}{t(P+P_4)} + (C-C_1)(D-D_1)(1+\beta-\beta_1) + \frac{(H-v_1)t}{2}\Big((1+\beta-\beta_1)\left(\big(1-(\beta+\beta_4)\big)(P-P_1)-(D+D_4)\right) + (\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big),$$

$$\mathcal{G}_2 = \frac{SD}{tP} + CD(1+\beta) + \frac{Ht}{2}\Big((1+\beta)\big((1-\beta)P-D\big) + \beta^2(P-D)\Big),$$

$$\mathcal{G}_3 = \frac{(S+S_4)(D+D_4)}{t(P-P_1)} + (C+C_4)(D+D_4)(1+\beta+\beta_4) + \frac{(H+v_4)t}{2}\Big((1+\beta+\beta_4)\left(\big(1-(\beta-\beta_1)\big)(P+P_4)-(D-D_1)\right) + (\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big).$$

The graded mean integration representation of the fuzzy number $\tilde{G}(t)$ is obtained from Equation (3.12) as

$$DF\left(\tilde{\mathcal{G}}(t)\right)=\frac{1}{6}\Bigg[\frac{(S-S_1)(D-D_1)}{t(P+P_4)}+(C-C_1)(D-D_1)(1+\beta-\beta_1)+\frac{(H-v_1)t}{2}\Big((1+\beta-\beta_1)\Big(\big(1-(\beta+\beta_4)\big)(P-P_1)-(D+D_4)\Big)+(\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big)\Bigg]+\frac{4}{6}\left[\frac{SD}{tP}+CD(1+\beta)+\frac{Ht}{2}\Big((1+\beta)\big((1-\beta)P-D\big)+\beta^2(P-D)\Big)\right]+\frac{1}{6}\Bigg[\frac{(S+S_4)(D+D_4)}{t(P-P_1)}+(C+C_4)(D+D_4)(1+\beta+\beta_4)+\frac{(H+v_4)t}{2}\Big((1+\beta+\beta_4)\Big(\big(1-(\beta-\beta_1)\big)(P+P_4)-(D-D_1)\Big)+(\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big)\Bigg]. \tag{4.1.31}$$

Proceeding as in the case Equation (4.1.14), the optimal production time is given by

$$t^*=\sqrt{\frac{2\left(\frac{(S-S_1)(D-D_1)}{(P+P_4)}+\frac{4SD}{P}+\frac{(S+S_4)(D+D_4)}{(P-P_1)}\right)}{F_1+4F_2^1+F_4}}, \tag{4.1.32}$$

where

$$F_2^1=H\Big(\big((1+\beta)\big((1-\beta)P-D\big)+\beta^2(P-D)\big)\Big).$$

4.2. Fuzzy production inventory model for fuzzy production time

In this subsection, the deterministic inventory model given in Equation (2.12) is fully fuzzified. That is, the input parameters$(S,D,P,C,\beta,H\,)$and the decision variable$(t)$are fuzzified. Let each input parameters and the decision variablebe positive trapezoidal fuzzy numbers as follows:

$\tilde{S}=(S-S_1,S-S_2,S+S_3,S+S_4),\widetilde{D}=(D-D_1,D-D_2,D+D_3,D+D_4)$
$\tilde{t}=(t-u_1,t-u_2,t+u_3,t+u_4),\tilde{P}=(P-P_1,P-P_2,P+P_3,P+P_4)$
$\tilde{C}=(C-C_1,C-C_2,C+C_3,C+C_4),\tilde{\beta}=(\beta-\beta_1,\beta-\beta_2,\beta+\beta_3,\beta+\beta_4),$

$$\widetilde{H}=(H-v_1,H-v_2,H+v_3,H+v_4).$$

Then, the total cost per unit time in fuzzy sensein Equation (2.12) is given as

$$\tilde{G}(\tilde{t})=\frac{\tilde{S}*\widetilde{D}}{\tilde{t}*\tilde{P}}+\tilde{C}*\widetilde{D}*\left(1+\tilde{\beta}\right)+\frac{\widetilde{H}*\tilde{t}}{2}\left\{\left(1+\tilde{\beta}\right)*\left(\left(1-\tilde{\beta}\right)*\tilde{P}-\widetilde{D}\right)+\tilde{\beta}^2*\left(\tilde{P}-\widetilde{D}\right)\right\} \tag{4.2.1}$$

where$+,\ -,\ *$ and/are the fuzzy arithmetical operations, andthe following additional equations can be written

$$\tilde{t} * \tilde{P} = \big((t-u_1)(P-P_1),(t-u_2)(P-P_2),(t+u_3)(P+P_3),(t+u_4)(P+P_4)\big) \quad (4.2.2)$$

$$\frac{\tilde{S} * \tilde{D}}{\tilde{t} * \tilde{P}} = \left(\frac{(S-S_1)(D-D_1)}{(t+u_4)(P+P_4)},\frac{(S-S_2)(D-D_2)}{(t+u_3)(P+P_3)},\frac{(S+S_3)(D+D_3)}{(t-u_2)(P-P_2)},\frac{(S+S_4)(D+D_4)}{(t-u_1)(P-P_1)}\right), \quad (4.2.3)$$

$$\frac{\tilde{H} * \tilde{t}}{2} = \left(\frac{(H-v_1)(t-u_1)}{2},\frac{(H-v_2)(t-u_2)}{2},\frac{(H+v_3)(t+u_3)}{2},\frac{(H+v_4)\big((t_1+u_4)\big)}{2}\right), \quad (4.2.4)$$

Substituting Equations.(4.1.2.), (4.1.5), (4.1.6) and (4.1.8)-(4.1.12), and (4.2.2)- (4.2.4) in Equation(4.2.1), then the fuzzy total inventory cost function is represented as a trapezoidal fuzzy number

$$\tilde{G}(\tilde{t}) = (G_1,G_2,G_3,G_4), \quad (4.2.5)$$

where

$$G_1 = \frac{(S-S_1)(D-D_1)}{(t+u_4)(P+P_4)} + (C-C_1)(D-D_1)(1+\beta-\beta_1) + \frac{(H-v_1)(t-u_1)}{2}\Big((1+\beta-\beta_1)\big(\big(1-(\beta+\beta_4)\big)(P-P_1) - (D+D_4)\big) + (\beta-\beta_1)^2\big(P-P_1-(D+D_4)\big)\Big),$$

$$G_2 = \frac{(S-S_2)(D-D_2)}{(t+u_3)(P+P_3)} + (C-C_2)(D-D_2)(1+\beta-\beta_2) + \frac{(H-v_2)(t-u_2)}{2}\Big((1+\beta-\beta_2)\big(\big(1-(\beta+\beta_3)\big)(P-P_2) - (D+D_3)\big) + (\beta-\beta_2)^2\big(P-P_2-(D+D_3)\big)\Big),$$

$$G_3 = \frac{(S+S_3)(D+D_3)}{(t-u_2)(P-P_2)} + (C+C_3)(D+D_3)(1+\beta+\beta_3) + \frac{(H+v_3)(t+u_3)}{2}\Big((1+\beta+\beta_3)\big(\big(1-(\beta-\beta_2)\big)(P+P_3) - (D-D_2)\big) + (\beta+\beta_3)^2\big(P+P_3-(D-D_2)\big)\Big)$$

and

$$G_4 = \frac{(S+S_4)(D+D_4)}{(t-u_1)(P-P_1)} + (C+C_4)(D+D_4)(1+\beta+\beta_4) + \frac{(H+v_4)\big((t+u_4)\big)}{2}\Big((1+\beta+\beta_4)\big(\big(1-(\beta-\beta_1)\big)(P+P_4) - (D-D_1)\big) + (\beta+\beta_4)^2\big(P+P_4-(D-D_1)\big)\Big).$$

The graded mean integration representation of the fuzzy number $\tilde{G}(\tilde{t})$ is obtained from Equation (3.11) as

$$DF\left(\tilde{G}(\tilde{t})\right) = \frac{1}{6}\left[\frac{(S-S_1)(D-D_1)}{t_4(P+P_4)} + (C-C_1)(D-D_1)(1+\beta-\beta_1) + \frac{(H-v_1)t_1}{2}\left((1+\beta-\beta_1)\left(\left(1-(\beta+\beta_4)\right)(P-P_1)-(D+D_4)\right) + (\beta-\beta_1)^2\left(P-P_1-(D+D_4)\right)\right)\right]$$

$$+\frac{2}{6}\left[\frac{(S-S_2)(D-D_2)}{t_3(P+P_3)} + (C-C_2)(D-D_2)(1+\beta-\beta_2) + \frac{(H-v_2)t_2}{2}\left((1+\beta-\beta_2)\left(\left(1-(\beta+\beta_3)\right)(P-P_2) - (D+D_3)\right) + (\beta-\beta_2)^2\left(P-P_2-(D+D_3)\right)\right)\right]$$

$$+\frac{2}{6}\left[\frac{(S+S_3)(D+D_3)}{t_2(P-P_2)} + (C+C_3)(D+D_3)(1+\beta+\beta_3) + \frac{(H+v_3)t_3}{2}\left((1+\beta+\beta_3)\left(\left(1-(\beta-\beta_2)\right)(P+P_3) - (D-D_2)\right) + (\beta+\beta_3)^2\left(P+P_3-(D-D_2)\right)\right)\right]$$

$$+\frac{1}{6}\left[\frac{(S+S_4)(D+D_4)}{t_1(P-P_1)} + (C+C_4)(D+D_4)(1+\beta+\beta_4) + \frac{(H+v_4)t_4}{2}\left((1+\beta + \beta_4)\left(\left(1-(\beta-\beta_1)\right)(P+P_4) - (D-D_1)\right) + (\beta+\beta_4)^2\left(P+P_4-(D-D_1)\right)\right)\right] \quad (4.2.6)$$

In order to find the parameters which minimize the defuzzified function $DF\left(\tilde{G}(\tilde{t})\right)$, we have to solve the following partial derivatives of $DF\left(\tilde{G}(\tilde{t})\right)$ with respect to $\tilde{t} = (t_1, t_2, t_3, t_4)$ each equal to zero, i.e.

$$\frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_1} = \frac{(H-v_1)\left((1+\beta-\beta_1)\left(\left(1-(\beta+\beta_4)\right)(P-P_1)-(D+D_4)\right) + (\beta-\beta_1)^2\left(P-P_1-(D+D_4)\right)\right)}{12} - \frac{(S+S_4)(D+D_4)}{6(P-P_1)t_1^2} = 0 \quad (4.2.7)$$

$$\frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_2} = \frac{(H-v_2)\left((1+\beta-\beta_2)\left((1-(\beta+\beta_3))(P-P_2)-(D+D_3)\right)+(\beta-\beta_2)^2\left(P-P_2-(D+D_3)\right)\right)}{6} - \frac{2(S+S_3)(D+D_3)}{6(P-P_2)t_2^2} = 0 \tag{4.2.8}$$

$$\frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_3} = \frac{(H+v_3)\left((1+\beta+\beta_3)\left((1-(\beta-\beta_2))(P+P_3)-(D-D_2)\right)+(\beta+\beta_3)^2\left(P+P_3-(D-D_2)\right)\right)}{6} - \frac{2(S-S_2)(D-D_2)}{6(P+P_3)t_3^2} = 0 \tag{4.2.9}$$

and

$$\frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_4} = \frac{(H+v_4)\left((1+\beta+\beta_4)\left((1-(\beta-\beta_1))(P+P_4)-(D-D_1)\right)+(\beta+\beta_4)^2\left(P+P_4-(D-D_1)\right)\right)}{12} - \frac{(S-S_1)(D-D_1)}{6(P+P_4)t_4^2} = 0 \tag{4.2.10}$$

Solving the above equations, we get

$$t_1 = \sqrt{\frac{2(S+S_4)(D+D_4)}{(P-P_1)(H-v_1)\left((1+\beta-\beta_1)\left((1-(\beta+\beta_4))(P-P_1)-(D+D_4)\right)+(\beta-\beta_1)^2\left(P-P_1-(D+D_4)\right)\right)}} \tag{4.2.11}$$

$$t_2 = \sqrt{\frac{2(S+S_3)(D+D_3)}{(P-P_2)(H-v_2)\left((1+\beta-\beta_2)\left((1-(\beta+\beta_3))(P-P_2)-(D+D_3)\right)+(\beta-\beta_2)^2\left(P-P_2-(D+D_3)\right)\right)}} \tag{4.2.12}$$

$$t_3 = \sqrt{\frac{2(S-S_2)(D-D_2)}{(P+P_3)(H+v_3)\left((1+\beta+\beta_3)\left((1-(\beta-\beta_2))(P+P_3)-(D-D_2)\right)+(\beta+\beta_3)^2\left(P+P_3-(D-D_2)\right)\right)}} \tag{4.2.13}$$

$$t_4 = \sqrt{\frac{2(S-S_1)(D-D_1)}{(P+P_4)(H+v_4)\left((1+\beta+\beta_4)\left((1-(\beta-\beta_1))(P+P_4)-(D-D_1)\right)+(\beta+\beta_4)^2\left(P+P_4-(D-D_1)\right)\right)}} \tag{4.2.14}$$

Note that, due to the construction of the defined fuzzy numbers we have $t_1 > t_2 > t_3 > t_4$, then the conditions$t_1 < t_2 < t_3 < t_4$are not satisfied. Then, we adopt the Lagrangean method described in Section 3.To do this, we convert the inequality constraint$t_2 - t_1 \geq 0$into equality constraint$t_2 - t_1 = 0$and minimizethe defuzzified function subject to$t_2 - t_1 = 0$.We have the Lagrangian function as

$$\mathcal{L}(t_1, t_2, t_3, t_4) = DF\left(\tilde{G}(\tilde{t})\right) - \lambda(t_2 - t_1), \tag{4.2.15}$$

Where $\lambda$is the Lagrangean multiplier.

Taking the partial derivatives of$\mathcal{L}(t_1, t_2, t_3, t_4)$ with respect to$t_1, t_2, t_3, t_4$and$\lambda$and putting the derivatives equal to zero, i.e.

$$\left.\begin{aligned} \frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_1} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_1} + \lambda = 0, \\ \frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_2} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_2} - \lambda = 0, \\ \frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_3} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_3} = 0, \\ \frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_4} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_4} = 0, \\ \frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial \lambda} &= -t_2 + t_1 = 0. \end{aligned}\right\} \tag{4.2.16}$$

Solving the above equality system, we have

$$t_1 = t_2 = \sqrt{\frac{2\left(\frac{(S+S_4)(D+D_4)}{(P-P_1)} + \frac{2(S+S_3)(D+D_3)}{(P-P_2)}\right)}{F_1 + 2F_2}}, \tag{4.2.17}$$

$$t_3 = \sqrt{\frac{2(S-S_2)(D-D_2)}{(P+P_3)(H+v_3)\left((1+\beta+\beta_3)\left(\left(1-(\beta-\beta_2)\right)(P+P_3)-(D-D_2)\right)+(\beta+\beta_3)^2\left(P+P_3-(D-D_2)\right)\right)}}, \tag{4.2.18}$$

$$t_4 = \sqrt{\frac{2(S-S_1)(D-D_1)}{(P+P_4)(H+v_4)\left((1+\beta+\beta_4)\left(\left(1-(\beta-\beta_1)\right)(P+P_4)-(D-D_1)\right)+(\beta+\beta_4)^2\left(P+P_4-(D-D_1)\right)\right)}}, \tag{4.2.19}$$

where

$$F_1 = (H-v_1)\Big((1+\beta-\beta_1)\left(\left(1-(\beta+\beta_4)\right)(P-P_1)-(D+D_4)\right) + (\beta-\beta_1)^2\left(P-P_1-(D+D_4)\right)\Big),$$

$$F_2 = (H-v_2)\Big((1+\beta-\beta_2)\left(\left(1-(\beta+\beta_3)\right)(P-P_2)-(D+D_3)\right) + (\beta-\beta_2)^2\left(P-P_2-(D+D_3)\right)\Big).$$

Since $t_1 = t_2 > t_3 > t_4$, the constraint $t_1 < t_2 < t_3 < t_4$ is not satisfied. We get the similar result if repeat the procedure by selecting any one of the other inequality constraints. Hence, we convert two of the inequality constraints $t_2 - t_1 \geq 0$ and $t_3 - t_2 \geq 0$ as equality and minimize the defuzzified function subject to $t_2 - t_1 = 0$ and $t_3 - t_2 = 0$. The Lagrangean function with multipliers λ_1 and λ_2 is given as

$$\mathcal{L}(t_1,t_2,t_3,t_4) = DF\left(\tilde{G}(\tilde{t})\right) - \lambda_1(t_2-t_1) - \lambda_2(t_3-t_2). \tag{4.2.20}$$

The solution is obtained by putting the partial derivatives of $\mathcal{L}(t_1,t_2,t_3,t_4)$ with respect to $t_1, t_2, t_3, t_4, \lambda_1$ and λ_2 are equate to zero, i.e.

$$\left.\begin{aligned}
\frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_1} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_1} + \lambda_1 = 0,\\
\frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_2} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_2} - \lambda_1 + \lambda_2 = 0,\\
\frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_3} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_3} - \lambda_2 = 0,\\
\frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial t_4} &= \frac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_4} = 0,\\
\frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial \lambda_1} &= -t_2 + t_1 = 0,\\
\frac{\partial \mathcal{L}(t_1,t_2,t_3,t_4)}{\partial \lambda_2} &= -t_3 + t_2 = 0.
\end{aligned}\right\} \tag{4.2.21}$$

Solving the above, we have

$$t_1 = t_2 = t_3 = \sqrt{\frac{2\left(\frac{(S+S_4)(D+D_4)}{(P-P_1)} + \frac{2(S+S_3)(D+D_3)}{(P-P_2)} + \frac{2(S-S_2)(D-D_2)}{(P+P_3)}\right)}{F_1 + 2F_2 + 2F_3}}, \tag{4.2.22}$$

$$t_4 = \sqrt{\frac{2(S-S_1)(D-D_1)}{(P+P_4)(H+v_4)\left((1+\beta+\beta_4)\left(\left(1-(\beta-\beta_1)\right)(P+P_4)-(D-D_1)\right)+(\beta+\beta_4)^2\left(P+P_4-(D-D_1)\right)\right)}}, \tag{4.2.23}$$

where

$$F_1 = (H-v_1)\Big((1+\beta-\beta_1)\left(\left(1-(\beta+\beta_4)\right)(P-P_1)-(D+D_4)\right) + (\beta-\beta_1)^2\left(P-P_1-(D+D_4)\right)\Big),$$

$$F_2 = (H-v_2)\Big((1+\beta-\beta_2)\left(\left(1-(\beta+\beta_3)\right)(P-P_2)-(D+D_3)\right) + (\beta-\beta_2)^2\left(P-P_2-(D+D_3)\right)\Big),$$

$$F_3 = (H+v_3)\Big((1+\beta+\beta_3)\left(\left(1-(\beta-\beta_2)\right)(P+P_3)-(D-D_2)\right) + (\beta+\beta_3)^2\left(P+P_3-(D-D_2)\right)\Big).$$

The results from Equations (4.2.22) and (4.2.23) show that$t_1 = t_2 = t_3 > t_4$, and then the condition$t_1 < t_2 < t_3 < t_4$is not satisfied. We get the similar result if repeat by selecting any two of the inequality constraints. Hence, the inequality constraints$t_2 - t_1 \geq 0$, $t_3 - t_2 \geq 0$and$t_4 - t_3 \geq 0$are converting into equalities as, $t_2 - t_1 = 0$ and $t_3 - t_2 = 0$and$t_4 - t_3 = 0$. The Lagrangean function with multipliers$\lambda_i, i = 1,2,3,4$ is

$$\mathcal{L}(t_1, t_2, t_3, t_4) = DF\left(\tilde{G}(\tilde{t})\right) - \lambda_1(t_2 - t_1) - \lambda_2(t_3 - t_2) - \lambda_3(t_4 - t_3). \tag{4.2.24}$$

In order to minimize the function$\mathcal{L}(t_1, t_2, t_3, t_4)$given in Equation (4.2.24), we take the partial derivatives of$\mathcal{L}(t_1, t_2, t_3, t_4)$with respect to$t_1, t_2, t_3, t_4, \lambda_1$, λ_2 and λ_3,and putting the resultsequal to zero. That is

$$\left.\begin{array}{c}
\dfrac{\partial \mathcal{L}(t_1, t_2, t_3, t_4)}{\partial t_1} = \dfrac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_1} + \lambda_1 = 0, \\
\dfrac{\partial \mathcal{L}(t_1, t_2, t_3, t_4)}{\partial t_2} = \dfrac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_2} - \lambda_1 + \lambda_2 = 0, \\
\dfrac{\partial \mathcal{L}(t_1, t_2, t_3, t_4)}{\partial t_3} = \dfrac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_3} - \lambda_2 + \lambda_3 = 0, \\
\dfrac{\partial \mathcal{L}(t_1, t_2, t_3, t_4)}{\partial t_4} = \dfrac{\partial DF\left(\tilde{G}(\tilde{t})\right)}{\partial t_4} - \lambda_3 = 0, \\
\dfrac{\partial \mathcal{L}(t_1, t_2, t_3, t_4)}{\partial \lambda_1} = -t_2 + t_1 = 0, \\
\dfrac{\partial \mathcal{L}(t_1, t_2, t_3, t_4)}{\partial \lambda_2} = -t_3 + t_2 = 0, \\
\dfrac{\partial \mathcal{L}(t_1, t_2, t_3, t_4)}{\partial \lambda_3} = -t_4 + t_3 = 0.
\end{array}\right\} \tag{4.2.25}$$

Thus, we have$t_1 = t_2 = t_3 = t_4 = t^*$, that is

$$t^* = \sqrt{\frac{2\left(\frac{(S+S_4)(D+D_4)}{(P-P_1)} + \frac{2(S+S_3)(D+D_3)}{(P-P_2)} + \frac{2(S-S_2)(D-D_2)}{(P+P_3)} + \frac{(S-S_1)(D-D_1)}{(P+P_4)}\right)}{F_1 + 2F_2 + 2F_3 + F_4}}, \tag{4.2.26}$$

where

$$F_1 = (H - v_1)\Big((1 + \beta - \beta_1)\Big(\big(1 - (\beta + \beta_4)\big)(P - P_1) - (D + D_4)\Big) + (\beta - \beta_1)^2\big(P - P_1 - (D + D_4)\big)\Big),$$

$$F_2 = (H - v_2)\Big((1 + \beta - \beta_2)\Big(\big(1 - (\beta + \beta_3)\big)(P - P_2) - (D + D_3)\Big) + (\beta - \beta_2)^2\big(P - P_2 - (D + D_3)\big)\Big),$$

$$F_3 = (H + v_3)\Big((1 + \beta + \beta_3)\Big(\big(1 - (\beta - \beta_2)\big)(P + P_3) - (D - D_2)\Big) + (\beta + \beta_3)^2\big(P + P_3 - (D - D_2)\big)\Big).$$

$$F_4 = (H + v_4)\Big((1 + \beta + \beta_4)\Big(\big(1 - (\beta - \beta_1)\big)(P + P_4) - (D - D_1)\Big) + (\beta + \beta_4)^2\big(P + P_4 - (D - D_1)\big)\Big).$$

In Equation (4.2.26), since the solution t^* satisfies all inequality constraints, t^* will be the optimum solution to the model. Since the optimal solution is the only one feasible solution, it is the optimum solution of the inventory model.

It is interesting to note that the optimal solution for the fuzzy model with fuzzy production time period given by Equation (4.2.26) is the same as the optimal solution for the fuzzy model with crisp production time period in Equation (4.1.17).

5. Numerical Examples

This section provides numerical examples to illustrate the behaviour of the fuzzy model and compare the results between the fuzzy case and crisp case using the parameters given by Jamal et al. (2004). Note that, Cárdenas-Barrón (2007) presented the correct solutions to the two numerical examples presented in Jamal et al. (2004).Consider an inventory model with crisp parameters having the following: $D = 300$units/year $P = 550$ units/year, $S = \$50/\text{batch}, H = \$50/\text{unit/year}, \beta = 0.05, C = \$7/\text{unit}$.

The optimal production time t_c^* and the total cost per unit time $TCU(t_c^*)$ of the crisp case can be derived from Equations. (2.12) and (2.14), respectively. Then, we get $t_c^* = 0.068$ years and $TCU(t_c^*) = \$3004.289$. Let t_{tri}^* and t_{tra}^* be the optimal production time periods and $DF(\tilde{G}(t_{tri}^*))$ and $DF(\tilde{G}(t_{tra}^*))$ be the defuzzified total cost functions for the triangular and trapezoidal fuzzy numbers, respectively. In Tables 1-3, we set some trapezoidal fuzzy numbers of the input parameters (S, D, P, C, β, H). For each of these parameters, the variations in the values are arranged arbitrary and their defuzzified values are determined by using the GMIR method and are shown in the second columns of the tables. Besides, the percentage difference between the fuzzy values and crisp case denoted by DF_k for the parameter k, which is also called the level of fuzziness, are also shown in the third and sixth columns of the tables.

In order to facilitate the computation process, the formulas were written in Microsoft Excel 2016. Based on these values, the results of the inventory policies for the proposed model for various sets of trapezoidal fuzzy numbers are given in Table 4.Table 4 indicates the optimal policy of the model presented in Section 4 and lists the optimum values of the total cost $DF(\tilde{G}(t_{tra}^*))$ and the production time t_{tra}^* computed from Equation (4.2.6) and Equation (4.2.26), respectively.

Table 1: Fuzzy trapezoidal values for the input parameters S and D.

$\tilde{S}$	$DF(\tilde{S})$	DF_S	$\tilde{D}$	$DF(\tilde{D})$	DF_D
(15,24.5,60,71)	42.5	-15%	(180,216,304,310)	255	-15%
(20,27,60.5,75)	45	-10%	(210,238.5,309,315)	270	-10%
(25,31,61,76)	47.5	-5%	(237,262.5,314,320)	285	-5%
(15,42.5,65,85)	52.5	+5%	(262,280,319,430)	315	+5%
(20,43,68,88)	55	+10%	(265,290,342.5,450)	330	+10%
(20,45,71,93)	57.5	+15%	(270,295,375,460)	345	+15%

Table 2: Fuzzy trapezoidal values for the input parameters P and C.

$\tilde{P}$	$DF(\tilde{P})$	DF_P	$\tilde{C}$	$DF(\tilde{C})$	DF_C
(340,345,555,665)	467.5	-15%	(1.7,2.5,8.5,12)	5.95	-15%
(345,367.5,600,690)	495	-10%	(1.8,2.55,8.95,13)	6.3	-10%
(365,370,650,730)	522.5	-5%	(2,3,9,13.9)	6.65	-5%
(475,478,651,732)	577.5	+5%	(2.5,3.5,10.05,14.5)	7.35	+5%
(500,510,685,740)	605	+10%	(3.6,3.8,10.2,14.6)	7.7	+10%
(510,525,700,835)	632.5	+15%	(3.5,4,11,14.8)	8.05	+15%

Table 3: Fuzzy trapezoidal values for the input parameters H and β.

$\tilde{H}$	$DF(\tilde{H})$	DF_H	$\tilde{\beta}$	$DF(\tilde{\beta})$	DF_β
(19,25,61,64)	42.5	-15%	(0.015,0.02,0.06,0.08)	0.0425	-15%
(20,30,62,66)	45	-10%	(0.016,0.021,0.0635,0.085)	0.045	-10%
(22,30,63,77)	47.5	-5%	(0.02,0.025,0.064,0.087)	0.0475	-5%
(28,35,68.5,80)	52.5	+5%	(0.022,0.03,0.07,0.093)	0.0525	+5%
(29,38,70,85)	55	+10%	(0.024,0.035,0.071,0.094)	0.055	+10%

(30,40,72.5,90)	57.5	+15%	(0.025,0.037,0.0755,0.095)	0.0575	+15%

Table 4 shows the changes in the optimal values of the decision variable and the total costs between fuzzy and crisp cases. The results indicate that the percentage changes for the input parameters is set to be from-15 to +15%corresponding to percentage difference in the defuzzified total cost value ranging from-9.26%to+51.19%. A -15% in the values of the input parameters results in an increase in the production time value of+4.13%,corresponding to a reduction in the total cost value of -9.26%, and an increase of+15% in the values of input parameters decreases the production time value by -10.36% but increases the total cost value by +51.19%.

Table 4: The change in optimal policy from the crisp case using the trapezoidal fuzzy numbers in Tables 1-3

t^*_{tra}	Change in t^*_{tra}(%)	$DF(\tilde{G}(t^*_{tra}))$	Change in $DF(\tilde{G}(t^*_{tra}))$ (%)
0.071	4.13	2,726.113	-9.26
0.069	1.67	2,942.069	-2.07
0.066	-2.89	3,180.418	+5.86
0.064	-6.44	3,800.775	+26.51
0.063	-8.25	4,127.905	+37.40
0.061	-10.36	4,542.223	+51.19

Variation of the percentage difference between the fuzzy case and crisp case is also illustrated in Figure 2. One can notice that while the values of production time decreases gradually, the total cost increases significantly,

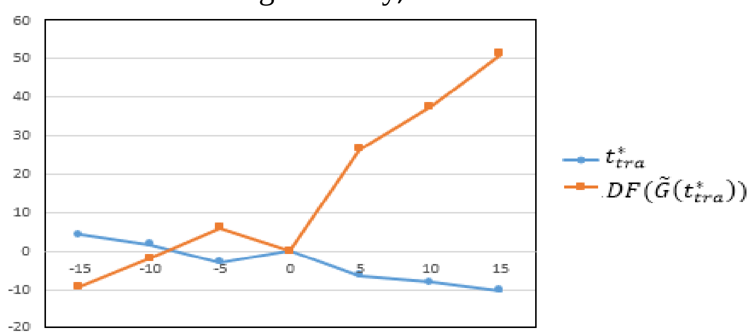

Figure2: Variation of the percentage difference DF_k effects on t^*_{tra} and $DF(\tilde{G}(t^*_{tra}))$.

The trapezoidal fuzzy numbers reduce to the triangular fuzzy numbers as shown in Tables 5 and 6 when the lower and upper modes are equal to the related crisp value.

Table 5: Triangular fuzzy numbers corresponding to trapezoidal fuzzy numbers

$\tilde{S}$	$\tilde{D}$	$\tilde{P}$	$\tilde{C}$

(15,50,71)	(180,300,310)	(340,550,665)	(1.7,7,12)
(20,50,75)	(210,300,315)	(345,550,690)	(1.8,7,13)
(25,50,76)	(237,300,320)	(365,550,730)	(2,7,13.9)
(15,50,85)	(262,300,430)	(475,550,732)	(2.5,7,14.5)
(20, 50,88)	(265,300,450)	(500,550,740)	(3.6,7,14.6)
(20, 50,93)	(270,300,460)	(510,550,835)	(3.5,7,14.8)

Table 6: Triangular fuzzy numbers corresponding to trapezoidal fuzzy numbers

$\widetilde{H}$	$\tilde{\beta}$
(19,501,64)	(0.015,0.05,0.08)
(20,50,66)	(0.016,0.05,0.085)
(22,50,77)	(0.02,0.05,0.087)
(28,50,80)	(0.022,0.05,0.093)
(29,50,85)	(0.024,0.05,0.094)
(30,50,90)	(0.025,0.05,0.095)

Table 7 shows the changes in the optimal values of the decision variable and the defuzzified total cost between the fuzzy and crisp cases. The results indicate that the percentage changes for the input parameters is set to be from -15%to +15% corresponding to percentage difference in the defuzzified total cost value ranging from+2.53 to+30.87%. A -10% in the values of the input parameters results in a decrease in the production time value of -1.12%,corresponding to an increase in the total cost value of+5.91%, and an increase of+10% in the values of input parameters decreases theproduction time value by -3.33% but increases the defuzzified total cost value by +27.09%.

Variation of the percentage difference between fuzzy case and crisp case is displayed in Figure 3. One notices that as DF_k increases, the value $DF(\tilde{G}(t_{tri}^*))$ increases gradually, but the value $DF(\tilde{G}(t_{tra}^*))$ increases importantly. Hence, the differences between $DF(\tilde{G}(t_{tra}^*))$and$DF(\tilde{G}(t_{tri}^*))$increases significantly asDF_kincreases.

Table 7: Optimal policy by using triangular fuzzy numbers

t^*_{tri}	Change in t^*_{tri}(%)	$DF(\tilde{G}(t^*_{tri}))$	Change in $DF(\tilde{G}(t^*_{tri}))$ (%)
0.0667	-2.2	3,080.172	+2.53
0.0675	-1.12	3,181,789	+5.91
0.0648	-5.0	3,301.616	+9.90
0.0664	-2.73	3,678.598	+22.44
0.0660	-3.33	3,818.299	+27.09
0.0628	-7.90	3,931.665	+30.87

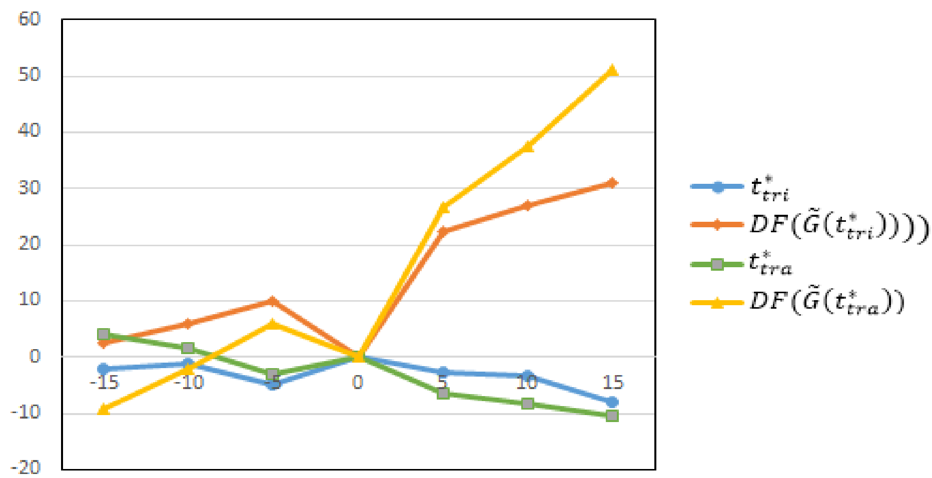

Figure3: The behavior of the total cost functions and the production times for fuzzy numbers with respect to the percentage difference between fuzzy case and crisp case.

It can be concluded from the results given above that the optimal values of t^* and its respective $DF(\tilde{G}(t^*))$, which is the total cost, changes with the variation in the fuzziness level in all model's components. Furthermore, it can be seen above that the variations in the component fuzziness levels' percentage highly affect the fuzzy total cost unlike in the case of the production time's optimal value. For the trapezoidal fuzzy numbers, it can be noticed that, while the range of the component fuzziness levels' percentage change is assigned from -15% to +15%, the percentage difference in $DF(\tilde{G}(t^*_{tra}))$ values is from -9.26% to +51.19%. For instance, -5% variation in the component fuzziness levels decreases t^*_{tra} value of -2.89% and increases the $DF(\tilde{G}(t^*_{tra}))$ value of +5.86%. On the other hand, with the variation of +15% in the component fuzziness levels, the relative change of the total cost will extend to +51.19%. Moreover, as it can be seen above that the fuzzy total cost changes in the same direction with the percentage increase in the component fuzziness levels, whereas the optimal production time slightly variates in the opposite direction.

The results shown above signify that if the variation in the component fuzziness levels reaches -15%, there will be lowest fuzzy total cost; however, there will be the lowest negative relative error with respect to the optimum values of crisp cases. Furthermore, due to the fuzziness in all the components, the percentage changes in the optimal values of the fuzzy models with triangular fuzzy numbers are less compared to the percentage changes in the case of the fuzzy models with trapezoidal fuzzy numbers. It is evident from all the tables that with the use of the trapezoidal fuzzy numbers, there will be minimum fuzzy total cost and a -15% decrease can be seen in the component fuzziness levels. In this specific case, the fuzzy EPQ model can have the following the optimal policy with rework options: optimal production time $t^* = 0.071$ years and fuzzy total cost $DF\left(\tilde{G}(t^*)\right) = \$2{,}726.113$.

6. Conclusions

Many applications of the concept of fuzziness in inventory problems have been studied by many researchers. The one that we have applied in this study is based on the fuzziness of the input parameters and the decision variables. In this paper; we have developed two production inventory models for a single-stage production system in fuzzy environment. In the first model, input parameters are considered as fuzzy numbers, while production time period is considered as crisp value. In the second model, both input parameters and the production time are considered as fuzzy numbers. Each fuzzy model are defuzzified using the graded mean integration representation (GMIR) method. These models are solved for triangular and trapezoidal fuzzy numbers. The optimal policy for the model with fuzzy production time is determined using Lagrangean optimization method. It can be observed that the optimal solution of using the fuzzy production inventory model with the fuzzy production time period is not distinct than that of using the fuzzy production inventory model with the crisp production time period; therefore, it is not significant to have the production time as a fuzzy number.

Numerical examples are provided for developed models, and effect of changes on optimal policy is studied. These examples showed that when the values of the production time decrease, the difference in the total cost between the fuzzy and crisp cases increases. This relationship may be beneficial to determine the simpler fuzzy inventory models, which the decision makers or researchers can use easily. The percentage change in the optimal production time is nearly equal of the changes in the input parameters. The results also signify that, by using triangular fuzzy numbers instead of trapezoidal fuzzy numbers, the total cost and the decision variable’s values are less affected due to the fuzziness in the components.

Future research study includes the investigation of a model where different defuzzification methods, such as signed distance and centroid method, are applied in order to give more realistic applications of fuzzy sets.

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