The Beta Generalized Inverse Weibull Geometric Distribution and its Applications

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Abstract

A new six-parameter distribution called the beta generalized inverse Weibull-geometric distribution is proposed. The new distribution is generated from the logit of a beta random variable and includes the generalized inverse Weibull geometric distribution. Various structural properties of the new distribution including explicit expressions for the moments, moment generating function, mean deviation are derived. The estimation of the model parameters is performed by maximum likelihood method.

Keywords: Generalized Inverse Weibull, Geometric distribution, Beta-G, Moments, Reliability Function, Maximum Likelihood.

1. Introduction

The inverse Weibull (IW) distribution has many applications in the reliability engineering discipline and model degradation of mechanical components such as the dynamic components (pistons, crankshafts of diesel engines, etc). It provides a good fit to several data such as the times to breakdown of an insulating fluid, subject to the action of constant tension. Also, it can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods, applications in medicine and ecology, determining the cost effectiveness, maintenance periods of reliability centered maintenance activities. Keller et al. (1985) obtained the IW model by investigating failures of mechanical components subject to degradation. de Gusmão et al. (2011) introduced the three-parameter generalized IW (GIW) distribution with decreasing and unimodal failure rate.

The cumulative distribution function (cdf) and probability density function (pdf) of the GIW distribution are defined (for $x \geq 0$) by

$$G(x,\alpha,\gamma,\theta) = e^{-\gamma(\alpha x)^{-\theta}} \quad \text{and} \quad g(x,\alpha,\gamma,\theta) = \alpha\theta\gamma(\alpha x)^{-\theta-1}e^{-\gamma(\alpha x)^{-\theta}}, \quad (1)$$

respectively, where $\alpha > 0$ is scale parameter and θ and γ are positive shape parameters.

Recently, there have been many attempts to define new families of probability distributions that extend well-known families of distributions and at the same time provide great flexibility in modeling data in practice. One such class of distributions generated by compounding the well-known lifetime distributions such as exponential, Weibull (W), generalized exponential and exponentiated W with the geometric (Gc) distribution. For example, Adamidis and Loukas (1998) introduced the exponential-geometric (EGc), Barreto-Souza et al. (2010) pioneered the Weibull-geometric (WGc) and Adamidis et al. (2005) proposed the extended exponential-geometric (EEGc) distributions.

Let N be a geometric random variable with probability mass function given by

$$P(n;p) = P(N = n) = (1-p)p^{n-1},\ \text{for} n \in N \text{and} p \in (0,1). \qquad (2)$$

In this paper, we define and study a new lifetime model called the *beta generalized inverse Weibull geometric* (BGIWGc) distribution. Its main characteristic is that three additional shape parameters are added in Equation (1) to provide more flexibility for the generated distribution. Based on compounding the GIW distribution with the Gc distribution and then using the beta-G (B-G) family pioneered by Eugene et al. (2002), we construct the six-parameter BGIWGc model and give a comprehensive description of some of its mathematical properties. We aim that it will attract wider applications in engineering, medicine and other areas of research.

At first, we will define the GIW-geometric (GIWGc) distribution and then we use the B-G to construct the BGIWGc model.

Suppose that a company has N systems functioning independently and producing a certain product at a given time, where N is a random variable, which is often determined by economy, customers demand, etc. The reason for considering N as a random variable comes from a practical viewpoint in which failure (of a device for example) often occurs due to the present of an unknown number of initial defects in the system. Let N has the pmf in (geometric).

Now, consider the failure times of the initial defects, denoted by $X_1, X_2, \dots, X_N$ be N independent and identically distributed (iid) random variables following the GIW distribution with cdf and pdf (1). Note that the time to failure of the first out of the N functioning systems is given by $X_{(1)} = \min\{X_i\}_{i=1}^{N}$. Then, the cdf of $X_{(1)}$ is given (for $x > 0$) by

$$G(x;\alpha,\theta,\gamma,p) = \frac{e^{-\gamma(\alpha x)^{-\theta}}}{1-p\left[1-e^{-\gamma(\alpha x)^{-\theta}}\right]}, \qquad (3)$$

where $p \in (0,1)$, $\alpha > 0, \beta > 0$ and $\gamma > 0$.

The corresponding pdf of the new model can be written as

$$g(x;\alpha,\theta,\gamma,p) = (1-p)\alpha\theta\gamma(\alpha x)^{-\theta-1}e^{-\gamma(\alpha x)^{-\theta}}\left\{1-p\left[1-e^{-\gamma(\alpha x)^{-\theta}}\right]\right\}^{-2} \qquad (4)$$

Henceforth, we denote a random variable X having pdf (4) by $X \sim \text{GIWGc}(\alpha, \theta, \gamma, p)$.

The survival function (sf) and hazard rate function (hrf) of the GIWGc distribution are given by

$$\overline{G}(x; \alpha, \theta, \gamma, p) = \frac{(1-p)e^{-\gamma(\alpha x)^{-\theta}}}{1 - pe^{-\gamma(\alpha x)^{-\theta}}}$$

and

$$h(x; \alpha, \theta, \gamma, p) = \frac{(1-p)\alpha\theta\gamma(\alpha x)^{-\theta-1}e^{-\gamma(\alpha x)^{-\theta}}}{\left[1 - p\left(1 - e^{-\gamma(\alpha x)^{-\theta}}\right)\right]\left[(1-p)e^{-\gamma(\alpha x)^{-\theta}}\right]}.$$

For an arbitrary baseline cdf $G(x; \phi)$ the B-G family due to Eugene et al. (2002) has the cdf and pdf given (for $x > 0$) by

$$F(x; a, b, \phi) = \frac{1}{B(a,b)} \int_0^{G(x;\phi)} w^{a-1}(1-w)^{b-1} dw = \frac{B_{G(x;\phi)}(a,b)}{B(a,b)} = I_{G(x;\phi)}(a,b), \quad (5)$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weight and

$$B_y(a,b) =_0^y w^{a-1}(1-w)^{b-1} dw$$

is the incomplete beta function with $B(a,b) = B_1(a,b)$ and $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$ is the incomplete beta function ratio. One major benefit of this class of distributions is its ability of fitting skewed data that cannot be properly fitted by existing distributions. If $b = 1$, $F(x) = G(x)^a$ and then F is usually called the exponentiated G distribution (or the Lehmann type-I distribution). Indeed, if Z is a beta distributed random variable with parameters a and b, then the cdf of $X = G^{-1}(Z)$ agrees with the cdf given in (5).

The B-G has the following special cases

• If $G(x; \phi)$ is the cdf of a standard uniform distribution, the cdf (5) yields the cdf of the beta distribution with parameters a and b.

• If a is an integer value and $b = n - a + 1$, then the cdf (5) becomes

$$F(x; a, b, \phi) = \frac{1}{B(a, n-a+1)} \int_0^{G(x;\phi)} w^{a-1}(1-w)^{b-1} dw$$
$$= \sum_{i=a}^{n} \binom{n}{i} [G(x; \phi)]^i [1 - G(x; \phi)]^{n-i},$$

which is really the cdf of the a_{th} order statistic of a random sample of size n from distribution $G(x; \phi)$.

• For $a = b = 1$, then the cdf (5) reduces to $F(x; \phi) = G(x; \phi)$.

• The cdf (5) reduces to $F(x; b, \phi) = [1 - G(x; \phi)]^b$, for $a = 1$.

• When $b = 1$, the cdf (5) reduces to $F(x; a, \phi) = [G(x; \phi)]^a$.

For general a and b, the cdf (5) can be defined in terms of the well-known hypergeometric function by

$$F(x;a,b,\phi)=\frac{1}{aB(a,b)}G(x;\phi)^{a}{}_2F_1\big(a,1-b,a+1;G(x;\phi)\big),$$

where

$${}_2F_1(\alpha,\theta,\gamma;x)=\sum_{i=0}^{\infty}\frac{(\alpha)_i(\theta)_i}{(\gamma)_i i!}x^i,\ |x|<1,$$

where $(\alpha)_i=\frac{\Gamma(\alpha+i)}{\Gamma(\alpha)}=\alpha(\alpha+1)\ldots(\alpha+i-1)$ denotes the ascending factorial of α. We obtain the properties of $F(x)$ for any B-G distribution defined from a parent $G(x;\phi)$ in (5) could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik (2000).

The pdf and hrf of the B-G class are given by

$$f(x;a,b,\phi)=\frac{g(x;\phi)}{B(a,b)}G(x;\phi)^{a-1}\{1-G(x;\phi)\}^{b-1} \tag{6}$$

and

$$h(x;a,b,\phi)=\frac{g(x;\phi)G(x;\phi)^{a-1}\{1-G(x;\phi)\}^{b-1}}{B(a,b)I_{[1-G(x;\phi)]}(a,b)},$$

respectively, where $I_{[1-G(x;\phi)]}(a,b)=1-I_{G(x;\phi)}(a,b)=\overline{F}(x;a,b,\phi)$ is the survival function of the B-G distribution.

The B-G class has been used to construct various extensions for the well-known distributions. For example, the beta normal (Eugene et al., 2002), beta Fréchet (Nadarajah and Gupta, 2004), beta exponential (Nadarajah and Kotz, 2006), beta W (Lee et al., 2007), beta W-geometric (Cordeiro et al., 2011), beta generalized exponential (Barreto-Souza et al., 2010), beta modified W (da Silva et al., 2010), beta IW (BIW) (Khan, 2010), beta exponential-geometric (Bidram, 2012) and beta exponential Fréchet (Mead et al., 2017) distributions.

The reminder of the paper is organized as follows. In Section 2, we define the BGIWGc distribution and its special cases. The expansion for its pdf and cdf are given in Section 3. Moments, moment generating function and mean deviation are discussed in Section 3. In Section 4 we obtain the Rényi and Shannon entropy of BGIWGc distribution. Finally, maximum likelihood estimation is performed in Section 5. The rest of the paper is organized as follows. In Section 2, we present the probability density function (PDF) and failure rate function and provide plots of such functions for selected parameter values. In Section 3, we obtain the moment generating and characteristic functions. We also give the moments of the order statistics. The Rényi entropy is derived in Section 4. Maximum likelihood estimation of the parameters and the expected information matrix are discussed in Section 5. Section 6 deals with the estimation of the stress strength parameter. An application of the GEG model to real data is illustrated in

2. The BGIWGc distribution

In this section, we difine the six parameter BGIWGc distribution by inserting (3) in equation (5). Then, the cdf of the BGIWGc distribution is given by

$$F(x;\varphi)=\frac{1}{B(a,b)}\int_0^{\frac{e^{-\gamma(\alpha x)^{-\theta}}}{1-p\left(1-e^{-\gamma(\alpha x)^{-\theta}}\right)}} w^{a-1}(1-w)^{b-1}dw\,, x>0, \qquad (7)$$

where $a>0$ and $b>0$ are shape parameters. The random variable X with the cdf (7) is said to have a BGIWGc distribution and will be denoted by $X\sim$BGIWGc(φ) where $\varphi=(\alpha,\gamma,\theta,p,a,b)$.

The corresponding pdf of (7) takes the form

$$f(x;\varphi)=\frac{(1-p)^b\alpha\theta\gamma(\alpha x)^{-\theta-1}e^{-\gamma a(\alpha x)^{-\theta}}\left(1-e^{-\gamma(\alpha x)^{-\theta}}\right)^{b-1}}{B(a,b)\left[1-p\left(1-e^{-\gamma(\alpha x)^{-\theta}}\right)\right]^{a+b}}. \qquad (8)$$

The hazard (failure) rate function (hrf) of X is given by

$$h(x;\varphi)=\frac{(1-p)^b\theta(\alpha x)^{-\theta-1}e^{-\gamma a(\alpha x)^{-\theta}}\left[1-e^{-\gamma(\alpha x)^{-\theta}}\right]^{b-1}}{B(a,b)\left\{1-p\left[1-e^{-\gamma(\alpha x)^{-\theta}}\right]\right\}^{a+b}I_{[1-G(x;\alpha,\gamma,\theta)]}(a,b)}.$$

The BGIWGc distribution is a very flexible model that approaches to different distributions when its parameters are changed. Its 27 sub-models are listed in Table 1. Figure 1 displays some plots of the BGIWGc pdf for some values and the plots of its hrf are shown in Figure 2.

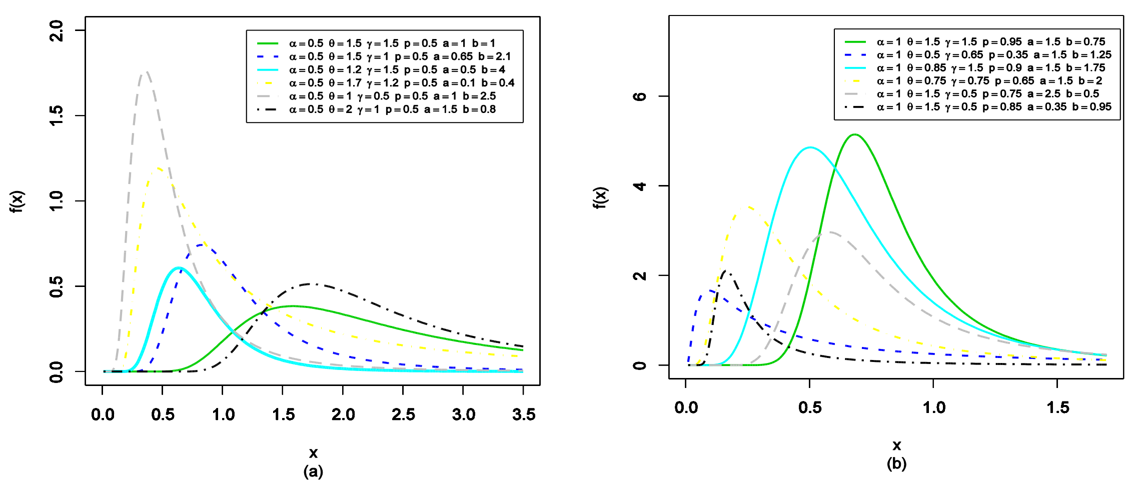

Figure 1: Pdf plots of the BGIWGc distribution for selected values of the parameters

3. Linear representation

In this subsection we present a useful linear representation for the BGIWGc pdf. Consider the series expantion given, for $|z| < 1$ and b is a positive real non-integer, by

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i) i!} z^i \tag{9}$$

Applying (9) to Equations (8), we can write

$$F(x,\varphi) = \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j) j!} \int_0^{\frac{e^{-\gamma(\alpha x)^{-\theta}}}{1-p(1-e^{-\gamma(\alpha x)^{-\theta}})}} w^{a+j-1} dw$$

$$= \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j) j! (a+j)} \left(\frac{e^{-\gamma(\alpha x)^{-\theta}}}{1-p(1-e^{-\gamma(\alpha x)^{-\theta}})} \right)^{a+j-1} .$$

Table 1: Sub-models of the BGIWGc distribution

No.	Distribution	α	γ	θ	p	a	b
1	BIWGc	α	1	θ	p	a	b
2	BIW	α	1	θ	$\downarrow 0$	a	b
3	BGIEGc	α	γ	1	p	a	b
4	BGIEGc	α	γ	1	p	a	b
5	BIEGc	α	1	1	p	a	b
6	BGIE	α	1	1	$\downarrow 0$	a	b
7	BGIRGc	α	γ	2	p	a	b
8	BIRGc	α	1	2	p	a	b
9	BGIR	α	1	2	$\downarrow 0$	a	b
10	BFrGc	1	1	θ	p	a	b
11	BFr	1	1	θ	$\downarrow 0$	a	b
12	BIEGc	1	1	1	p	a	b
13	BIRGc	1	1	2	p	a	b
14	BIE	α	1	1	$\downarrow 0$	a	b
15	BIR	α	1	2	$\downarrow 0$	a	b
16	GIWGc	α	γ	θ	p	1	1
17	IWGc	α	1	θ	p	1	1
18	GIEGc	α	γ	1	p	1	1
19	IEGc	α	1	1	p	1	1
20	GIRGc	α	γ	2	p	1	1
21	IRGc	α	1	2	p	1	1
22	GIW	α	γ	θ	$\downarrow 0$	1	1
23	GIE	α	γ	1	$\downarrow 0$	1	1
24	GIR	α	γ	2	$\downarrow 0$	1	1
25	IW	α	1	θ	$\downarrow 0$	1	1
26	IE	α	1	1	$\downarrow 0$	1	1
27	IR	α	1	2	$\downarrow 0$	1	1

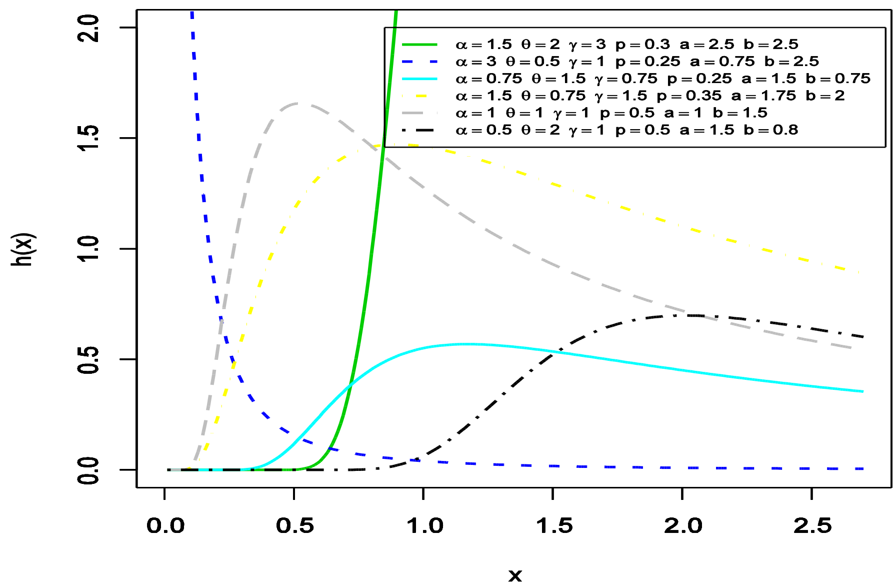

Figure 2: Hrf plots of the BGIWGc distribution for selected values of the parameters

Using the series representation

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{\Gamma(k)i!} z^i, |z| < 1, k > 0. \quad (10)$$

The last pdf can be expressed, after applying (10), as

$$f(x,\varphi) = \frac{(1-p)^b p^j}{\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(b+j)}{j!} g_j\big(x, \alpha, \theta, \gamma, a, (b+j)\big), \quad (11)$$

where $g_j(x, \alpha, \theta, \gamma, a, (b+j))$ is the pdf of the BGIW distribution with parameters $\alpha, \theta, \gamma, a$ and $(b+j)$. Equation (11) reveals that the BGIWGc dnsity function can be expressed as an infinite linear mixture of BGIW densities. Hence, we can obtain some mathematical properties of the BGIWGc distribution from those properties of the BGIW distribution.

3. Properties

In this section, we study the statistical properties of the BGIWGc distribution including ordinary and incomplete moments and moment generating function.

3.1 Ordinary and incomplete moments

In this sub- section,we derive the expression for ordinary and incomplete moments of BGIWGc . The moments of different orders will help in determining the expected life time of a device and also the dispersion, skewness and kurtosis in a given set of observations arising in reliability applications. rth moment on the origin of X can be obtained using the well-known formula.

Lemma (3.1): If X has BGIWGc(φ) where $\varphi = (\alpha, \gamma, \theta, p, a, b)$ then the r_{th} moment of $X, r = 1,2, \ldots$. has the following form:

$$\mu'_r = \sum_{k=0}^{\infty} \psi_k \frac{\gamma \alpha^{-r} \Gamma(1-\frac{r}{\theta})}{[\gamma(a+k)]^{1-\frac{r}{\theta}}}, \tag{12}$$

where

$$\psi_k = \sum_{j=0}^{\infty} \frac{(1-p)^b p^j}{\Gamma(b)\Gamma(b)} (-1)^k \binom{b+j}{k}.$$

Proof.

Let X be a random variable with density function (8). The rth ordinary moment of the BGIWGc distribution is given by

$$\mu'_r = \int_0^{\infty} x^r f(x, \phi) dx$$
$$= \frac{(1-p)^b p^j}{\Gamma(b)\Gamma(b)} \sum_{j,k=0}^{\infty} (-1)^k \binom{b+j}{k} \alpha\theta\gamma \int_0^{\infty} x^r (\alpha x)^{-\theta-1} e^{-\gamma(a+k)(\alpha x)^{-\theta}} dx$$
$$= \frac{(1-p)^b p^j}{\Gamma(b)\Gamma(b)} \sum_{k,j=0}^{\infty} (-1)^k \binom{b+j}{k} \frac{\gamma \alpha^{-r} \Gamma(1-\frac{r}{\theta})}{[\gamma(a+k)]^{1-\frac{r}{\theta}}} = \sum_{k=0}^{\infty} \psi_k \frac{\gamma \alpha^{-r} \Gamma(1-\frac{r}{\theta})}{[\gamma(a+k)]^{1-\frac{r}{\theta}}}.$$

which completes the proof.

Lemma (3.2): If X has BGIWGc(ϕ), then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{r,k=0}^{\infty} \psi_k \frac{t^r}{r!} \frac{\gamma \alpha^{-r} \Gamma(1-\frac{r}{\theta})}{[\gamma(a+k)]^{1-\frac{r}{\theta}}}. \tag{13}$$

Proof.

We start with the well known definition of the moment generating function given by $M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x, \phi) dx$, Since $\sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x)$ converges and each term is integrable for all t close to 0, then we can rewrite the moment generating function as $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$ by replacing $E(X^r)$. Hence using (12) the MGF of BGIWGc distribution is given by

$$M_X(t) = \sum_{r,k=0}^{\infty} \psi_k \frac{t^r}{r!} \frac{\gamma \alpha^{-r} \Gamma(1-\frac{r}{\theta})}{[\gamma(a+k)]^{1-\frac{r}{\theta}}}.$$

which completes the proof .

Similarly, the characteristic function of the BGIWGc distribution becomes $\varphi_X(t) = M_X(it)$ where $i = \sqrt{-1}$ is the unit imaginary number.

3.2 Conditional moments

For lifetime models, it is also of interest to find the conditional moments and the mean residual lifetime function. The conditional moments for BGIWGc distribution is given by the following Lemma**.**

Lemma (3.3): If X has BGIWGc(ϕ), the conditional moments for BGIWGc distribution is given by

$$E(X^s|X>t)=\sum_{k=0}^{\infty}\psi_k\left[\frac{\gamma\alpha^{-r}\Gamma(1-\frac{r}{\theta},\gamma(a+k)(\alpha t)^{-\theta})}{[\gamma(a+k)]^{1-\frac{r}{\theta}}}\right].$$

Proof.

$$E(X^s|X>t)=\int_t^{\infty}x^s f(x,\phi)dx$$

$$=\frac{(1-p)^b p^j}{\Gamma(b)\Gamma(b)}\sum_{j,k=0}^{\infty}(-1)^k\binom{b+j}{k}\alpha\theta\gamma\int_t^{\infty}x^s(\alpha x)^{-\theta-1}e^{-\gamma(a+k)(\alpha x)^{-\theta}}dx$$

$$=\sum_{k=0}^{\infty}\psi_k\left[\frac{\gamma\alpha^{-s}\Gamma(1-\frac{s}{\theta},\gamma(a+k)(\alpha t)^{-\theta})}{[\gamma(a+k)]^{1-\frac{s}{\theta}}}\right],$$

where $\Gamma(s,t)=\int_t^{\infty}x^{s-1}e^{-x}dx$ is the upper incomplete gamma function.

The mean residual lifetime function of beta additive Weibull distribution is given by

$$\mu(t)=E(X|X>t)-t=\frac{1}{\overline{F}(t)}\sum_{k=0}^{\infty}\psi_k\left[\frac{\gamma\alpha^{-1}\Gamma(1-\frac{1}{\theta},\gamma(a+k)(\alpha t)^{-\theta})}{[\gamma(a+k)]^{1-\frac{s}{\theta}}}\right]-t.$$

The importance of the MRL function is due to its uniquely determination of the lifetime distribution as well as the hrf.

3.3 Residual and reversed residual functions

Given that a component survives up to time $t\geq 0$, the residual life is the period beyond t until the time of failure and defined by the conditional random variable $X-t|X>t$. In reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely. Therefore, the rth-order moment of the residual lifetime can be obtained via the general formula

$$\mu_r(t)=E((X-t)^r|X>t)=\frac{1}{\overline{F}(t)}\int_t^{\infty}(x-t)^r f(x,\varphi)dx, r\geq 1.$$

Applying the binomial expansion of $(x-t)^r$ into the above formula, we have

$$\mu_r(t)=\frac{1}{\overline{F}(t)}\sum_{k=0}^{\infty}\sum_{d=0}^{r}\psi_k(-t)^d\binom{r}{d}\theta\gamma\,\alpha^{-\theta-1}\int_t^{\infty}x^{r-d-\theta-1}e^{-\gamma(a+k)(\alpha x)^{-\theta}}dx$$

$$=\frac{1}{\overline{F}(t)}\sum_{k=0}^{\infty}\sum_{d=0}^{r}\psi_k(-t)^d\binom{r}{d}\frac{\gamma\Gamma(1-\frac{r-d}{\theta},\gamma(a+k)(\alpha t)^{-\theta})}{\alpha^{r-\theta}[\gamma(a+k)]^{1-\frac{r-d}{\theta}}}.$$

On the other hand, we analogously discuss the reversed residual life and some of its properties. The reversed residual life can be defined as the conditional random variable $t - X|X \leq t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t.

The rth-order moment of the reversed residual life can be obtained as

$$m_r(t) = E((t-X)^r|X \leq t) = \frac{1}{F(t)}\int_t^{\infty}(t-x)^r f(x,\varphi)dx, r \geq 1.$$

Using (8) and applying the binomial expansion of $(t-x)^r$, we can write

$$m_r(t) = \frac{1}{F(t)}\sum_{k=0}^{\infty}\sum_{d=0}^{r}\psi_k(-t)^d\binom{r}{d}\theta\gamma\,\alpha^{-\theta-1}\int_t^{\infty}x^{r-d-\theta-1}e^{-\gamma(a+k)(\alpha x)^{-\theta}}dx$$
$$= \frac{1}{F(t)}\sum_{k=0}^{\infty}\sum_{d=0}^{r}\psi_k(-t)^d\binom{r}{d}\frac{\gamma\zeta(1-\frac{r-d}{\theta},\gamma(a+k)(\alpha t)^{-\theta})}{\alpha^{r-\theta}[\gamma(a+k)]^{1-\frac{r-d}{\theta}}},$$

where $\zeta(s,t) = \int_0^{\mathrm{t}} x^{s-1}e^{-x}dx$ is the lower incomplete gamma function.

3.4 Mean deviations and Bonferroni and Lorenz curves

The mean deviation about the mean and mean deviation about the median measure the amount of scatter in a population. For random variable X with pdf $f(x)$, cdf $F(x)$, mean $\mu = E(X)$ and $M =$ Median (X), the mean deviation about the mean and mean deviation about the median are, respectively, defined by

$$\delta_1(x) = \int_0^{\infty}|x-\mu|f(x)dx = 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty}xf(x)dx$$

and

$$\delta_2(x) = \int_0^{\infty}|x-M|\,f(x)dx = -\mu + 2\int_M^{\infty}xf(x)dx.$$

Then, for X, we have

$$\int_{\mu}^{\infty}xf(x)dx = \sum_{k=0}^{\infty}\psi_k\left[\frac{\gamma\alpha^{-s}\Gamma(1-\frac{s}{\theta},\gamma(a+k)(\alpha\mu)^{-\theta})}{[\gamma(a+k)]^{1-\frac{s}{\theta}}}\right]$$

and

$$\int_M^{\infty}xf(x)dx = \sum_{k=0}^{\infty}\psi_k\left[\frac{\gamma\alpha^{-s}\Gamma(1-\frac{s}{\theta},\gamma(a+k)(\alpha M)^{-\theta})}{[\gamma(a+k)]^{1-\frac{s}{\theta}}}\right].$$

Using the last two integrals, one can obtain $\delta_1(x)$ and $\delta_2(x)$.

The Bonferroni and Lorenz Curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined by

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx=\frac{1}{p\mu}\sum_{k=0}^{\infty}\psi_k\left[\frac{\gamma\alpha^{-s}\zeta(1-\frac{s}{\theta},\gamma(a+k)(\alpha q)^{-\theta})}{[\gamma(a+k)]^{1-\frac{s}{\theta}}}\right]$$

and

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx=\frac{1}{\mu}\sum_{k=0}^{\infty}\psi_k\left[\frac{\gamma\alpha^{-s}\zeta(1-\frac{s}{\theta},\gamma(a+k)(\alpha q)^{-\theta})}{[\gamma(a+k)]^{1-\frac{s}{\theta}}}\right].$$

4. Maximum likelihood estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the BGIWGc distribution from complete samples only. Let $x_1,\dots,x_n$ be a random sample of size n from the BGIWGc distribution given by (8).

The total log-likelihood function for φ, where $\varphi=(\alpha,\theta,\gamma,p,a,b)^T$, is given by

$$\begin{aligned}\ell_n=\ell_n(\varphi)&=nb\log(1-p)+n\log(a+b)-n\log a-n\log b\\&+n\log\gamma-(\theta+1)\sum_{i=1}^{n}\log(\alpha x_i)-\gamma a\sum_{i=1}^{n}(\alpha x_i)^{-\theta}\\&+n\log\alpha+(b-1)\sum_{i=1}^{n}\log\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)\\&+n\log\theta-(a+b)\sum_{i=1}^{n}\log\left[1-p\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)\right].\end{aligned}$$

The log-likelihood can be maximized either directly by using the SAS program or R-language or by solving the nonlinear likelihood equations obtained by differentiating the last equation.

The components of the score function $U_n(\varphi)=\left(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\theta},\frac{\partial\ell_n}{\partial\gamma},\frac{\partial\ell_n}{\partial p},\frac{\partial\ell_n}{\partial a},\frac{\partial\ell_n}{\partial b}\right)^T$ are

$$\begin{aligned}\frac{\partial\ell_n}{\partial\alpha}&=\frac{n}{\alpha}-(\theta+1)\sum_{i=1}^{n}\left(\frac{1}{\alpha}\right)+\gamma a\theta\sum_{i=1}^{n}x_i(\alpha x_i)^{-\theta-1}\\&-(b-1)\sum_{i=1}^{n}\frac{\gamma a\theta x_i(\alpha x_i)^{-\theta-1}e^{-\gamma(\alpha x_i)^{-\theta}}}{1-e^{-\gamma(\alpha x_i)^{-\theta}}}\\&-p(a+b)\sum_{i=1}^{n}\frac{\gamma a\theta x_i(\alpha x_i)^{-\theta-1}e^{-\gamma(\alpha x_i)^{-\theta}}\left(1-e^{-\gamma(\alpha x)^{-\theta}}\right)}{\left[1-p\left(1-e^{-\gamma(\alpha x)^{-\theta}}\right)\right]}=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell_n}{\partial\theta}&=\frac{n}{\theta}-\sum_{i=1}^{n}\log(\alpha x_i)\sum_{i=1}^{n}\ln(\alpha x_i)+\gamma\sum_{i=1}^{n}(\alpha x_i)^{-\theta}\ln(\alpha x_i)\\&-(b-1)\sum_{i=1}^{n}\frac{\gamma(\alpha x_i)^{-\theta}\ln(\alpha x_i)e^{-\gamma(\alpha x_i)^{-\theta}}}{\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)}\\&-p(a+b)\sum_{i=1}^{n}\frac{\gamma(\alpha x_i)^{-\theta}\ln(\alpha x_i)e^{-\gamma(\alpha x_i)^{-\theta}}\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)}{\left[1-p\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)\right]}=0,\end{aligned}$$

$$\frac{\partial \ell_n}{\partial \gamma} = \frac{n}{\gamma} - a\sum_{i=1}^{n}(\alpha x_i)^{-\theta} + (b-1)\sum_{i=1}^{n}\frac{(\alpha x_i)^{-\theta}e^{-\gamma(\alpha x_i)^{-\theta}}}{\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)}$$

$$+p(a+b)\sum_{i=1}^{n}\frac{(\alpha x_i)^{-\theta}e^{-\gamma(\alpha x_i)^{-\theta}}}{\left[1-p\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)\right]} = 0,$$

$$\frac{\partial \ell_n}{\partial p} = \frac{-nb}{p} + (a+b)\sum_{i=1}^{n}\frac{\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)}{\left[1-p\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)\right]} = 0,$$

$$\frac{\partial \ell_n}{\partial a} = -\psi(a) + \psi(a,b) - \sum_{i=1}^{n}\log\left[1-p\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right)\right] = 0$$

and

$$\frac{\partial \ell_n}{\partial b} = n\log(1-p) - \psi(b) + \psi(a,b) + \sum_{i=1}^{n}\log\left(1-e^{-\gamma(\alpha x_i)^{-\theta}}\right) = 0.$$

The maximum likelihood estimation (MLE) of φ, say $\hat{\varphi}$, is obtained by solving the nonlinear system $U_n(\varphi) = 0$. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimate $\hat{\varphi}$. For interval estimation and hypothesis tests on the model parameters, we require the information matrix.

Applying the usual large sample approximation, MLE of φ, i.e $\hat{\varphi}$ can be treated as being approximately $N_6(\varphi, J_n(\varphi)^{-1})$, where $J_n(\varphi) = E[I_n(\varphi)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\varphi} - \varphi)$ is $N_6(0, J(\varphi)^{-1})$, where $J(\varphi) = \lim_{n\to\infty} n^{-1} I_n(\varphi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\varphi)$ is replaced by the average sample information matrix evaluated at $\hat{\varphi}$, say $n^{-1}I_n(\hat{\varphi})$. The estimated asymptotic multivariate normal $N_6(\varphi, I_n(\hat{\varphi})^{-1})$ distribution of $\hat{\varphi}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1-\gamma)$ asymptotic confidence interval for each parameter φ_r is given by

$$ACI_r = \left(\hat{\varphi}_r - z_{\frac{\gamma}{2}}\sqrt{\widehat{I_{rr}}}, \hat{\varphi}_r + z_{\frac{\gamma}{2}}\sqrt{\widehat{I_{rr}}}\right)$$

where z_γ is the upper $100\gamma th$ percentile of the standard normal distribution.

5. Data analysis

In this section, we provide an application to a real data set to assess the performance and flexibility of the BGIWGc distribution. In order to compare the new model with other fitted distributions, we consider some goodness-of-fit statistics including $-2\hat{\ell}$, Anderson-Darling statistic (A^*) and Cramér-von Mises statistic (W^*), where $\hat{\ell}$ denotes the maximized log-likelihood, Generally, the smaller these statistics are, the better the fit.

The data set (Gross and Clark, 1975, page 105) on the relief times of twenty patients receiving an analgesic is: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. For these data, we compare the fits of the BGIWGc distribution with the beta transmuted W (BTW) (Afify et al., 2017), McDonald log-logistic (McLL) (Tahir et al., 2014), McDonald Weibull (McW) (Cordeiro et al., 2014), new modified W (NMW) (Almalki and Yuan, 2013), transmuted complementary W-geometric (TCWG) (Afify et al., 2014), beta W (BW) (Lee et al., 2007) and exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al., 2015) distributions. The pdfs of these distributions are given in Appendix A.

Tables 2 list the values of $-2\hat{\ell}$, W^* and A^* whereas the MLEs of the model parameters and their corresponding standard errors are given in Table 3.

Table 2 compares the fits of the BGIWGc distribution with the BTW, McLL, McW, NMW, TCWG, BW and ETGR distributions. The figures in these tables show that the BGIWGc distribution has the lowest values for $-2\hat{\ell}$, W^* and A^* statistics among all fitted distributions. So, it could be chosen as the best model. The fitted pdf and QQ plot for the BGIWGc distribution are displayed in Figure 3. Figure 4 shows the estimated cdf and sf for the BGIWGc model. It is evident from these plots that the new model provides close fit to the data.

Table 2: Goodness-of-fit statistics for the relief times data

Model	$-2\hat{\ell}$	W^*	A^*
BGIWGc	31.662	0.0434	0.24665
BTW	33.051	0.06896	0.39769
McLL	33.854	0.07904	0.46199
McW	33.907	0.08021	0.46927
NMW	41.173	0.17585	1.0678
TCWG	33.607	0.07252	0.43603
BW	34.396	0.0873	0.51316
ETGR	36.856	0.13629	0.79291

Table 3: MLEs and their standard errors for the relief times data

Model	Estimates (standard errors)					
BGIWGc $(\alpha,\gamma,\theta,p,a,b)$	19.1874 (33.03)	20.5968 (43.241)	1.4346 (0.837)	9.8485 (2.001)	$39.2308\cdot 10^{-5}$ (63.252)	5.8015 (4.346)
BTW $(\alpha,\beta,a,b,\lambda)$	5.6186 (9.353)	0.5311 (0.148)	53.3438 (111.453)	3.5683 (4.265)	-0.7718 (3.894)	
McLL (α,β,a,b,c)	0.8811 (0.109)	2.0703 (3.693)	19.2254 (22.341)	32.0332 (43.077)	1.9263 (5.165)	
McW (α,β,a,b,c)	2.7738 (6.38)	0.3802 (0.188)	79.108 (119.131)	17.8976 (39.511)	3.0063 (13.968)	
NMW $(\alpha,\beta,\gamma,\delta,\theta)$	0.1215 (0.056)	2.7837 (20.37)	$8.2272\cdot 10^{-5}$ $(1.512\cdot 10^{-3})$	0.0003 (0.025)	2.7871 (0.428)	
TCWG $(\alpha,\beta,\gamma,\lambda)$	43.6627 (45.459)	5.1271 (0.814)	0.2823 (0.042)	-0.2713 (0.656)		
BW (α,β,a,b)	0.8314 (0.954)	0.6126 (0.34)	29.9468 (40.413)	11.6319 (21.9)		
ETGR $(\alpha,\beta,\lambda,\delta)$	0.1033 (0.436)	0.6917 (0.086)	-0.342 (1.971)	23.5392 (105.137)		

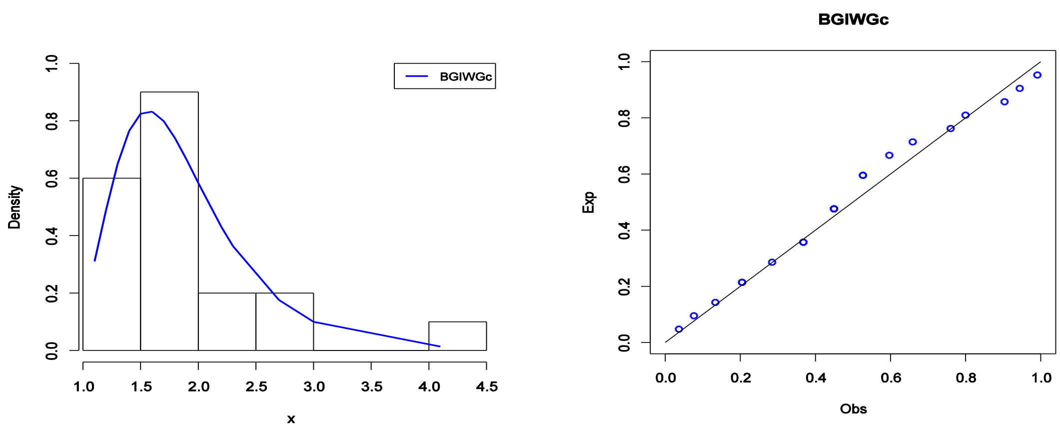

Figure 3: The fitted pdf and QQ plot of the BGIWGc model

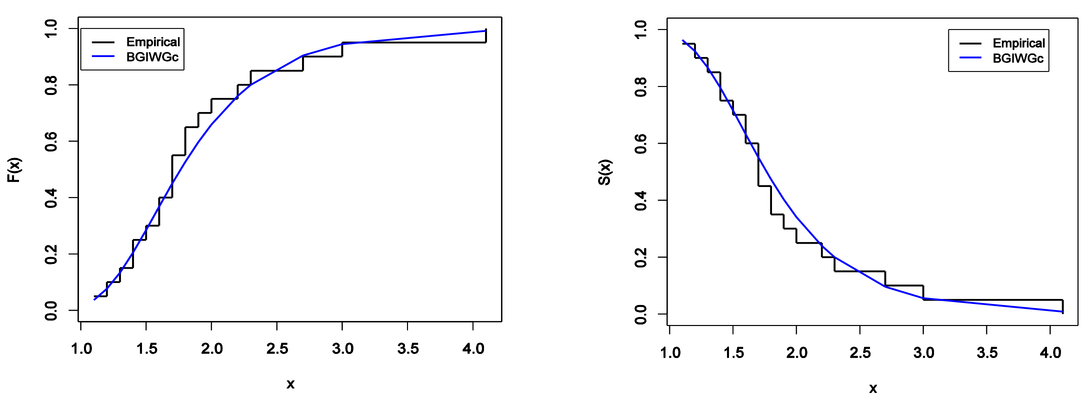

Figure 4: The estimated cdf and sf of the BGIWGc model

6. Conclusions

In this paper, we propose a new six-parameter model called the beta generalized inverse Weibull-geometric (BGIWGc) distribution, which extends the generalized inverse Weibull-geometric (GIWGc) distribution. The BGIWGc density function can be expressed as a linear mixture of GIWGc densities. We derive explicit expressions for some of its mathematical properties. We discuss maximum likelihood estimation. The proposed distribution provides better fits than some other competitive models using a real data set. We hope that the proposed model will attract wider applications in applied areas such as lifetime analysis, hydrology, reliability, engineering.

Appendix A:

The pdfs of the fitted distributions are given (for $x > 0$) by

BTW: $f(x) = \frac{\beta\alpha^\beta}{B(a,b)} x^{\beta-1}\left[1-\lambda+2\lambda e^{-(\alpha x)^\beta}\right]\left\{\left[1-e^{-(\alpha x)^\beta}\right]\left[1+\lambda e^{-(\alpha x)^\beta}\right]\right\}^{a-1}$

$$\times e^{-(\alpha x)^\beta}\left\{1-\left[1-e^{-(\alpha x)^\beta}\right]\left[1+\lambda e^{-(\alpha x)^\beta}\right]\right\}^{b-1};$$

McLL: $f(x) = \frac{\alpha c}{B(a/c,b)\beta}\left(\frac{x}{\beta}\right)^{a\alpha-1}\left[1+\left(\frac{x}{\beta}\right)^\alpha\right]^{-a-1}\left(1-\left\{1-\left[1+\left(\frac{x}{\beta}\right)^\alpha\right]^{-1}\right\}^c\right);$

McW: $f(x) = \frac{\beta c\alpha^\beta}{B(a/c,b)}\ x^{\beta-1}\ \mathrm{e}^{-(\alpha x)^\beta}\left[1-\mathrm{e}^{-(\alpha x)^\beta}\right]^{a-1}\left\{1-\left[1-\mathrm{e}^{-(\alpha x)^\beta}\right]^c\right\}^{b-1};$

NMW: $f(x) = \left[\alpha\theta x^{\theta-1}+\gamma(\beta+\delta x)x^{\beta-1}\mathrm{e}^{\delta x}\right]\mathrm{e}^{-\left(\alpha x^\theta+\gamma x^\beta e^{\delta x}\right)};$

TCWG: $f(x) = \alpha\beta\gamma(\gamma y)^{\beta-1}\ \mathrm{e}^{-(\gamma y)^\beta}\left[\alpha(1-\lambda)-(\alpha-\alpha\lambda-\lambda-1)\ \mathrm{e}^{-(\gamma y)^\beta}\right]$

$$\times\left[\alpha+(1-\alpha)\ \mathrm{e}^{-(\gamma y)^\beta}\right]^{-3};$$

ETGR: $f(x) = 2\alpha\delta\beta^2\ x\ \mathrm{e}^{-(\beta x)^2}\left\{1+\lambda-2\lambda\left[1-\mathrm{e}^{-(\beta x)^2}\right]^\alpha\right\}$

$$\times\left[1-\mathrm{e}^{-(\beta x)^2}\right]^{\alpha\delta-1}\left\{1+\lambda-\lambda\left[1-\mathrm{e}^{-(\beta x)^2}\right]^\alpha\right\}^{\delta-1}.$$

BW: $f(x) = \frac{\beta\alpha^\beta}{B(a,b)} x^{\beta-1}\mathrm{e}^{-b(\alpha x)^\beta}\left[1-\mathrm{e}^{-(\alpha x)^\beta}\right]^{a-1};$

The parameters of the above pdfs are all positive real numbers except for the TCWG and ETGR distributions for which $|\lambda| \le 1$.

References

1. Adamidis, K., Dimitrakopoulou, T. and Loukas, S. (2005). On an extension of the exponential-geometric distribution. Statistics & probability letters, 73, 259-269.

2. Adamidis, K. and Loukas, S. (1998). A lifetime distribution with decreasing failure rate. Statistics and Probability Letters, 39, 35-42.

3. Afify, A. Z., Nofal, Z. M. and Butt, N. S. (2014). Transmuted complementary Weibull geometric distribution. Pakistan Journal of Statistics and Operations Research, 10, 435-454.

4. Afify, A. Z., Nofal, Z. M. and Ebraheim, A. N. (2015). Exponentiated transmuted generalized Rayleigh distribution: A new four parameter Rayleigh distribution. Pakistan Journal of Statistics and Operations Research, 11, 115-134.
5. Afify A. Z., Yousof, H. M. and Nadarajah, S. (2017). The beta transmuted-H family for lifetime data. Statistics and Its Interface, 10, 505-520.
6. Almalki, S. J. and Yuan, J. (2013). A new modified Weibull distribution. Reliability Engineering and System Safety, 111, 164-170.
7. Barreto-Souza, W., Santos A. H. S., Cordeiro, G. M. (2010). The beta generalized exponential distribution, Journal of Statistical Computation and Simulation, 80, 159-172.
8. Bidram. H (2012). The Beta Exponential-Geometric Distribution. Communications in Statistics- Simulation and Computation, 41, 1606-1622.
9. Cordeiro, G. M., Hashimoto, E. M. and Ortega, E. (2014). The McDonald Weibull model. Statistics, 48, 256-278.
10. Eugene, N., Lee, C., Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and Methods, 31, 497-512.
11. de Gusmão, F. R., Ortega, E. M. M. and Cordeiro, G. M. (2011). The generalized inverse Weibull distribution. Statistical Papers, 52, 591-619.
12. Gradshteyn, I. S. and Ryzhik, I. M. (2000). Table of Integrals, Series and Products (sixth edition).San Diego: Academic Press.
13. Gross, A. J. and Clark, V. A. (1975). Survival distributions: Reliability Applications in the Biomedical Sciences. John Wiley and Sons, New York.
14. Khan, M. S. (2010). The beta inverse Weibull distribution. International Transactions in Mathematical Sciences and Computer, 3, 113-119.
15. Keller, A. Z., Goblin, M. T. and Farnworth, N. R. (1985). Reliability analysis of commercial vehicle engines. Reliability Engineering, 10, 15-25.
16. Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: Some properties and applications to censored data. Journal of Modern Applied Statistical Methods, 6, 173-186.
17. Mead, M. E., Afify, A. Z., Hamedani, G. G. and Ghosh, I. (2017). The beta exponential Fréchet distribution with applications. Austrian Journal of Statistics, 46, 41-63.
18. Nadarajah, S. and Gupta, A.K. (2004). The beta Fréchet distribution. Far East Journal of Theoretical Statistics,14, 15-24.
19. Nadarajah, S., Kotz, S. (2006). The beta exponential distribution. Reliability Engineering and System Safety, 91, 689-697.
20. Tahir, M. H., Mansoor, M., Zubair, M. and Hamedani, G. (2014). McDonald log-logistic distribution with an application to breast cancer data. Journal of Statistical Theory and Applications, 13, 65-82.