Inference on Constant-Stress Accelerated Life Testing Based on Geometric Process for Extension of the Exponential Distribution under Type-II Progressive Censoring

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Abstract

In this paper, the geometric process is introduced as a constant-stress accelerated model to analyze a series of life data that obtained from several increasing stress levels. The geometric process (GP) model is assumed when the lifetime of test units follows an extension of the exponential distribution. Based on progressive censoring, the maximum likelihood estimates (MLEs) and Bayes estimates (BEs) of the model parameters are obtained. Moreover, a real dataset is analyzed to illustrate the proposed procedures. In addition, the approximate, bootstrap and credible confidence intervals (CIs) of the estimators are constructed. Finally, simulation studies are carried out to investigate the precision of the MLEs and BEs for the parameters involved.

Keywords: Geometric process; Accelerated life testing; Progressive censoring; Bayes estimation; Extension of the exponential distribution; Bootstrap confidence interval; Credible confidence interval; Simulation study.

1. Introduction

The aim of traditional life testing and reliability experiments is to analyze data of failure time that obtained under normal operating conditions. However, such life data is so difficult to collect as a result of the limited testing time and the high reliability of the products such as electronics, power cables, insulating materials, laser, engines, etc. Therefore, accelerated life testing (ALT) is one of the most common approaches that used to obtain enough failure data in a short period of time. In such testing, products are tested at higher than usual levels of stress (e.g., temperature, voltage, humidity, vibration or pressure) to induce early failures. The life data collected from such accelerated tests is then analyzed and extrapolated to estimate the life characteristics under normal operating conditions. The stress loading in ALT can be applied in different ways. Commonly used methods are constant-stress and step-stress. Nelson (1990) discussed the advantages and disadvantages of each of such methods.

In constant-stress ALT, each unit is run at constant high stress either the failure occurs, or the test is terminated. Constant-stress models were studied by several authors; see Kim and Bai (2002), Watkins and John (2008) and Abdel-Hamid (2009). Jaheen et al. (2014) considered the constant partially ALT under progressive censoring for generalized exponential distribution. Guan et al. (2014) obtained the optimal constant-stress accelerated life tests under complete sampling for the generalized exponential distribution. Mohie El-Din et al. (2016a) considered the constant-stress ALT for the extension of the exponential distribution under progressive censoring. Mohie El-Din et al. (2017a) obtained the optimal constant-stress accelerated life tests under complete sampling for Lindley distribution. Mohie El-Din et al. (2017b) extended the results of Mohie El-Din et al. (2016a) to progressive-stress ALT. Abd El-Raheem (2019) discussed the problem of the optimal plan of constant stress ALT for the extension of the exponential distribution under complete sampling. Abd El-Raheem (2019b) expanded his results in Abd El-Raheem (2019a) to the censored case.

In step-stress ALT, the stress on each unit is not constant but is increased step by step at prespecified times or upon the occurrence of a fixed number of failures. The step-stress models were studied extensively in the literature; see Miller and Nelson (1983), Bai et al. (1989) and Gouno et al. (2004). Balakrishnan et al. (2007) considered the simple step-stress ALT under type-II censoring, assuming a cumulative exposure model for exponential distribution. Mohie El-Din et al. (2015a) applied the simple step-stress ALT under progressive first-failure censoring, considering a tampered random variable model for Weibull distribution. Mohie El-Din et al. (2015b) developed Bayes estimation for step-stress ALT to power generalized Weibull distribution under progressive censoring, using a tampered random variable model. Mohie El-Din et al. (2016b) considered the step-stress ALT for the extension of the exponential distribution under progressive censoring.

The concept of the geometric process was introduced by Lam (1988), in repair replacement problems. Since it was introduced, many studies in system reliability have shown that the geometric process model is an efficient and simple model for data analysis with a single trend or multiple trends. For example, see Lam and Zhang (1996) and Zhang (2008). So far, few studies utilize the geometric process in the analysis of accelerated life testing. Huang (2011) considered the GP model in ALT with complete and censored exponential samples. Kamal et al. (2012) extended the GP model for complete Weibull failure data. Zhou et al. (2012) applied the GP in constant-stress ALT based on the progressive type-I hybrid censoring for Rayleigh failure data.

In 2011, Nadarajah and Haghighi (2011) introduced an extension of the exponential distribution as an alternative to the gamma, Weibull and exponentiated exponential distributions. It provides great flexibility to analyze any real positive data. It has

increasing as well as decreasing failure rates depending on the values of the shape parameter. It has increasing (decreasing) failure rate function when $\gamma > 1\,(\gamma < 1)$ respectively.
The extension of the exponential distribution EE (γ,σ) is specified by the probability density function (pdf):

$$f(t) = \gamma\sigma(1+\sigma t)^{\gamma-1} \exp\left\{1-(1+\sigma t)^{\gamma}\right\}, \; t > 0, \gamma > 0, \sigma > 0, \qquad (1.1)$$

the corresponding cumulative distribution function (cdf) is given by

$$F(t) = 1 - exp\left\{1-(1+\sigma t)^{\gamma}\right\}, \qquad t > 0, \gamma > 0, \sigma > 0, \qquad (1.2)$$

and the corresponding hazard rate function (hrf) is given by

$$h(t) = \gamma\sigma(1+\sigma t)^{\gamma-1}. \qquad (1.3)$$

For $\gamma = 1$, the pdf in (1.1) reduces to the pdf of the exponential distribution.

The paper is drafted as follows: In Section 2, a description of the acceleration model and test assumptions are presented. In Section 3, the MLEs of the model parameters are derived. The BEs under square error loss function and LINEX loss function of model parameters are obtained in Section 4. In Section 5, a real dataset is analyzed to illustrate the proposed procedures in Sections 3 and 4. In Section 6, the asymptotic, bootstrap and credible confidence bounds for the model parameters are constructed. Section 7, contains the simulation studies. The conclusion is made in Section 8.

2. Model description

In 1988, Lam (1988) defined geometric process as follows:

Definition 2.1 *A sequence of nonnegative random variables* $\{X_n, n = 1,2,...\}$ *is said to be a geometric process, if they are independent and the distribution function of* X_n *is given by* $F(a^{n-1}x)$ *for* $n = 1,2,...,$ *where* F *is the distribution function of* X_1 *and* $a > 0$ *is called the ratio of the GP.*

It is clear to see that a GP is stochastically increasing if $0 < a < 1$, and it is stochastically decreasing if $a > 1$. Therefore, the GP is a natural approach to analyze data from a series of events with trend. It can be shown that if $\{X_n, n = 1,2,...\}$ is a GP and the pdf of X_1 is f with mean μ and variance σ^2, then the pdf of X_n is given by $a^{n-1} f(a^{n-1}x)$, with $E(X_n) = \dfrac{\mu}{a^{n-1}}$ and $Var(X_n) = \dfrac{\sigma^2}{a^{2(n-1)}}$, for more details see Lam (2007).

In the following, we illustrate how stochastically decreasing geometric process can serve as accelerated model. For this purpose, we consider two groups of assumptions: group A for the constant-stress model and group B for the geometric process model and then we prove that two groups are equivalent.

Group A:

1. Under constant-stress level ϕ_k, $k = 0,1,...,s$, the failure time T_k follows EE (γ, σ_k) distribution.

2. The scale parameter σ is a log-linear function of stress ϕ that is

   $\log \sigma_k = \alpha_0 + \alpha_1 \phi_k, \qquad k = 0,1,2,...,s,$

   where α_0 and $\alpha_1 > 0$ are unknown parameters depending on the nature of the product.

3. The values of stress are equidistant that is $\phi_k = \phi_0 + k\,d,\ \ k = 0,1,2,...,s,$ where d is the stress increment.

Group B

1. Assume $\{T_k, k = 0,1,2,...,s\}$ forms a geometric process, with ratio $a > 1$.
2. The failure time T_0 follows EE (γ, σ_0) distribution.

Lemma 2.1 *If the values of stress are equidistant that is, $\phi_k = \phi_0 + k\,d,\ \ k = 0,1,2,...,s,$ where d is the stress increment, then the life characteristic $\{\sigma_k, k = 0,1,2,...,s\}$ forms a geometric sequence with the ratio $a = e^{\alpha_1 d}$.*

Proof. From the assumptions 2 and 3 in group A, we get

$$\frac{\sigma_k}{\sigma_{k-1}} = e^{\alpha_1 d}, \qquad k = 1,...,s, \tag{2.1}$$

equation (2.1) shows that when the increased stress levels $\{\phi_0, \phi_1, \phi_2, ..., \phi_s\}$ forms an arithmetic sequence with a constant difference d, the life characteristic $\{\sigma_k, k = 0,1,2,...,s\}$ forms a geometric sequence with the ratio $a = e^{\alpha_1 d} > 1$.

Theorem 2.1 *If $\{T_k, k = 0,1,2,...,s\}$ forms a stochastically decreasing geometric process, then the life characteristic $\{\sigma_k, k = 0,1,2,...,s\}$ forms a geometric sequence.*

Proof. Since $\{T_k, k = 0,1,2,...,s\}$ is a geometric process, then

$$F_{T_k}(t) = F_{T_0}(a^k t), \qquad k = 0,1,...,s,\ a > 1,$$

from assumption 2 in group B, we get

$$F_{T_0}(t) = 1 - exp\left\{1 - \left(1 + \sigma_0 t\right)^{\gamma}\right\},$$

thus,

$$F_{T_k}(t) = 1 - exp\left\{1 - \left(1 + a^k \sigma_0 t\right)^{\gamma}\right\},$$

then T_k, $k = 0,1,...,s,$ has EE distribution with scale parameter $\sigma_k = a^k \sigma_0$, then

$$\frac{\sigma_k}{\sigma_{k-1}} = a, \qquad k = 1,...,s,\, a > 1, \tag{2.2}$$

equation (2.2) shows that the life characteristic $\{\sigma_k, k = 0,1,2,...,s\}$ forms a geometric sequence with the ratio $a > 1$.

It is evident from Lema 2.1 and Theorem 2.1 that the two groups of assumptions A and B are equivalent. Therefore, a stochastically decreasing geometric process can serve as accelerated model.

Theorem 2.2 *In constant-stress ALT, if the values of stress are equidistant, then the lifetimes under each stress level form a geometric process. That is, if* $\phi_k = \phi_0 + k\,d$, $k = 0,1,2,...,s$, *where* d *is the stress increment, then* $\{T_k, k = 0,1,2,...,s\}$ *forms a geometric process.*

Proof. From the assumptions 2 and 3 in group A, we get

$$\frac{\sigma_k}{\sigma_{k-1}} = e^{\alpha_1 d}, \qquad k = 1,2,...,s, \tag{2.3}$$

thus,

$$\sigma_k = a^k \sigma_0, \tag{2.4}$$

where σ_0 is the scale parameter of EE distribution under use-stress level ϕ_0, and $a = e^{\alpha_1 d} > 1$ is the acceleration factor from ϕ_0 to ϕ_1.

From the assumption 1 in group A and equation (2.4), then the pdf of the failure time of an item under the stress level ϕ_k, $k = 0,1,2,...,s$, is given by

$$\begin{aligned} f_{T_k}(t) &= a^k \gamma \sigma_0 \left(1 + a^k \sigma_0 t\right)^{\gamma-1} \exp\left\{1 - \left(1 + a^k \sigma_0 t\right)^{\gamma}\right\} \\ &= a^k f_{T_0}(a^k t), \qquad\qquad k = 0,1,...,s, \end{aligned} \tag{2.5}$$

where $f_{T_0}(t)$ is the pdf at use-stress level ϕ_0.

From (2.5), then the cdf of the failure time of an item under the stress level ϕ_k, $k = 0,1,2,...,s$, is given by

$$\begin{aligned} F_{T_k}(t) &= 1 - exp\left\{1 - \left(1 + a^k \sigma_0 t\right)^{\gamma}\right\}, \\ &= F_{T_0}(a^k t), \qquad k = 0,1,...,s, \end{aligned} \tag{2.6}$$

where $F_{T_0}(t)$ is the cdf at use-stress level ϕ_0.

Then $\{T_0, T_1, T_2, ..., T_s\}$ forms a geometric process with the ratio $a = e^{\alpha_1 d} > 1$.

According to constant ALT, suppose that there are s increasing stress levels and under each level n_k items are inspected, where k is the index of stress level, $k = 1,2,...,s$. Under each stress level, progressive type-II censoring is applied as follows: under the stress level k, $k=1,2,..,s,$ at the time of the first failure t_{k1}, $R_{k,1}$ items are randomly withdrawn from the remaining n_k-1 surviving items. At the time of the second failure t_{k2}, $R_{k,2}$ items from the remaining $n_k-2-R_{k,1}$ items are randomly withdrawn. The test terminates at the time of m_k-th failure occurs t_{km_k}, at this time all remaining $R_{k,m_k}=n_k-m_k-\sum_{i=1}^{m_k-1}R_{k,i}$ items are withdrawn. It is clear that the complete samples and type-II censored samples are special cases of this scheme. For more details about progressive type-II censoring, see Balakrishnan and Aggarwala (2000), and Balakrishnan and Cramer (2014). With these notations the observed progressive censored data under the k-th stress level is $t_{k1}<t_{k2}<...<t_{km_k}, k=1,2,...,s.$

3. Maximum likelihood estimation

Based on progressive type-II censored samples $T_{k1}<T_{k2}<...<T_{km_k}$, with censoring schemes $(R_{k,1},R_{k,2},...,R_{k,m_k})$, $k=1,2,...,s,$ the MLEs of the model parameters are obtained. Let the observed data under the stress level ϕ_k is $t_{k1}<t_{k2}<...<t_{km_k}, k=1,2,...,s,$ then the likelihood function of γ, σ_0 and a is given by

$$L(\gamma,\sigma_0,a)=\prod_{k=1}^{s}\left\{C_k\prod_{i=1}^{m_k}f_{T_k}(t_{ki})[1-F_{Tk}(t_{ki})]^{R_{k,i}}\right\}, \tag{3.1}$$

where

$$C_k=n_k\left(n_k-1-R_{k,1}\right)\left(n_k-2-R_{k,1}-R_{k,2}\right)...\left(n_k-m_k+1-\sum_{i=1}^{m_k-1}R_{k,i}\right).$$

From (2.5) and (2.6) in (3.1), we get

$$L(\gamma,\sigma_0,a)=\prod_{k=1}^{s}\left\{C_k\prod_{i=1}^{m_k}a^k\sigma_0\gamma(\omega(t_{ki}))^{\gamma-1}\exp\left\{(R_{k,i}+1)\left(1-(\omega(t_{ki}))^{\gamma}\right)\right\}\right\}, \tag{3.2}$$

where $\omega(t_{ki})=(1+a^k\sigma_0 t_{ki})$, the log-likelihood function may then be written as

$$\begin{aligned}\lambda(\gamma,\sigma_0,a) &=\sum_{k=1}^{s}\log C_k+\log(a)\sum_{k=1}^{s}km_k+(\log(\gamma)+\log(\sigma_0))\sum_{k=1}^{s}m_k\\ &+\sum_{k=1}^{s}\sum_{i=1}^{m_k}\left[(R_{k,i}+1)\left(1-(\omega(t_{ki}))^{\gamma}\right)+(\gamma-1)\log\left(\omega(t_{ki})\right)\right],\end{aligned} \tag{3.3}$$

the likelihood equations of γ, σ_0 and a are respectively

$$\frac{\partial\lambda}{\partial\gamma}=\frac{\sum_{k=1}^{s}m_k}{\gamma}+\sum_{k=1}^{s}\sum_{i=1}^{m_k}\left[\log\left(\omega(t_{ki})\right)\left(1-(R_{k,i}+1)\,(\omega(t_{ki}))^{\gamma}\right)\right], \tag{3.4}$$

$$\frac{\partial\lambda}{\partial\sigma_0}=\frac{\sum_{k=1}^{s}m_k}{\sigma_0}+\sum_{k=1}^{s}\sum_{i=1}^{m_k}a^k\left[\psi(t_{ki})\,(\omega(t_{ki}))^{\gamma-1}-\phi(t_{ki})\right], \tag{3.5}$$

$$\frac{\partial \lambda}{\partial a} = \frac{\sum_{k=1}^{s} k m_k}{a} + \sum_{k=1}^{s}\sum_{i=1}^{m_k} k a^{k-1}\sigma_0\left[\phi(t_{ki}) - \psi(t_{ki})\,(\omega(t_{ki}))^{\gamma-1}\right], \tag{3.6}$$

where $\phi(t_{ki}) = \dfrac{(\gamma-1)t_{ki}}{(1+a^k\sigma_0 t_{ki})}$, and $\psi(t_{ki}) = \gamma(R_{k,i}+1)t_{ki}$.

Now, we have a system of three nonlinear equations in three unknowns γ, σ_0 and a. It is clear that a closed form solutions are very difficult to obtain. Therefore, an iterative procedure such as Newton Raphson can be used to find numerical solutions of the above nonlinear system.

4. Bayes estimation

Square error (SE) loss function and linear exponential (LINEX) loss function are considered to obtain BEs of the model parameters $\gamma,\ \sigma_0$ and a under progressive type-II censoring. Assume that the model parameters $\gamma,\ \sigma_0$ and a are independent with priors as follows:

$$\pi(\gamma) \propto \gamma^{\mu-1} e^{-\gamma/\lambda}, \quad \gamma > 0, \mu, \lambda > 0,$$

$$\pi(\sigma_0) \propto e^{-\sigma_0/\beta}, \quad \sigma_0 > 0, \beta > 0,$$

$$\pi(a) \propto \frac{1}{a}, \quad a > 1,$$

then, the joint prior of the parameters $\gamma,\ \sigma_0$ and a is given by

$$\pi_1(\gamma, \sigma_0, a) \propto \frac{\gamma^{\mu-1}}{a} e^{-(\frac{\gamma}{\lambda}+\frac{\sigma_0}{\beta})}, \quad \gamma, \sigma_0 > 0, a > 1. \tag{4.1}$$

The joint posterior density function of the parameters $\gamma,\ \sigma_0$ and a can be written from (3.2) and (4.1) as follows:

$$\begin{aligned} \pi_1^*(\gamma, \sigma_0, a) &\propto L(\gamma, \sigma_0, a)\, \pi_1(\gamma, \sigma_0, a) \\ &\propto \frac{\gamma^{\mu-1}}{a} e^{-(\frac{\gamma}{\lambda}+\frac{\sigma_0}{\beta})} \prod_{k=1}^{s}\prod_{i=1}^{m_k} a^k \sigma_0 \gamma (\omega(t_{ki}))^{\gamma-1} \exp\left\{(R_{k,i}+1)\left(1-(\omega(t_{ki}))^{\gamma}\right)\right\}. \end{aligned} \tag{4.2}$$

Based on SE loss function and LINEX loss function, the Bayes estimator of the function of parameters $U(\Theta) = U(\gamma, \sigma_0, a)$ is respectively

$$\tilde{U}_{SE}(\Theta) = E(U(\Theta)), \tag{4.3}$$

and

$$\tilde{U}_{LINEX}(\Theta) = -\frac{1}{c}\log[E(e^{-cU(\Theta)})], \tag{4.4}$$

where $E(.)$ is the expected value and $c \neq 0$ is the shape parameter of LINEX loss function.

Unfortunately, we cannot compute these expectations explicitly. Therefore, Markov chain Monte Carlo (MCMC) method is used to approximate these expectations.

4.1 Bayesian estimation using MCMC method

In this subsection, MCMC method is applied to generate samples from the posterior distribution and then compute the BEs of γ, σ_0 and a.

From the joint posterior density function in (4.2), the conditional posterior distributions of γ, σ_0 and a are given respectively by

$$\pi_1^*(\gamma \mid \sigma_0, a) \propto \gamma^{\mu-1} e^{-\gamma/\lambda} \prod_{k=1}^{s} \prod_{i=1}^{m_k} \gamma (\omega(t_{ki}))^{\gamma} \exp\left\{ (R_{k,i}+1)\left(1-(\omega(t_{ki}))^{\gamma}\right) \right\}, \tag{4.5}$$

$$\pi_1^*(\sigma_0 \mid \gamma, a) \propto \sigma_0^{\sum_{k=1}^{s} m_k} e^{-\sigma_0/\beta} \prod_{k=1}^{s} \prod_{i=1}^{m_k} (\omega(t_{ki}))^{\gamma-1} \exp\left\{ (R_{k,i}+1)\left(1-(\omega(t_{ki}))^{\gamma}\right) \right\}, \tag{4.6}$$

$$\pi_1^*(a \mid \gamma, \sigma_0) \propto a^{\sum_{k=1}^{s} k m_k - 1} \prod_{k=1}^{s} \prod_{i=1}^{m_k} (\omega(t_{ki}))^{\gamma-1} \exp\left\{ (R_{k,i}+1)\left(1-(\omega(t_{ki}))^{\gamma}\right) \right\}. \tag{4.7}$$

The conditional posterior distributions of γ, σ_0 and a in (4.5), (4.6) and (4.7) cannot be reduced analytically to well known distributions. Therefore, the Metropolis method with normal proposal distribution is used to generate random samples from these distributions, see Metropolis et al. (1953).

The following algorithm is proposed to generate γ, σ_0 and a from the conditional posterior distributions and then obtain the BEs.

Algorithm (1)

1. Start with $\gamma^{(0)} = \hat{\gamma}_{MLE}$, $\sigma_0^{(0)} = \hat{\sigma}_{0MLE}$, $a^{(0)} = \hat{a}_{MLE}$.
2. Set $i = 1$.
3. Generate γ^* from proposal distribution $N(\gamma^{(i-1)}, var(\gamma^{(i-1)}))$.
4. Calculate the acceptance probability

$$r(\gamma^{(i-1)} \mid \gamma^*) = \min\left[1, \frac{\pi_1^*(\gamma^* \mid {\sigma_0}^{(i-1)}, a^{(i-1)})}{\pi_1^*(\gamma^{(i-1)} \mid {\sigma_0}^{(i-1)}, a^{(i-1)})}\right].$$

5. Generate U from $U(0,1)$.
6. If $U \le r(\gamma^{(i-1)} \mid \gamma^*)$, accept the proposal distribution and set $\gamma^{(i)} = \gamma^*$. Otherwise, reject the proposal distribution and set $\gamma^{(i)} = \gamma^{(i-1)}$.
7. To generate σ_0^*, do the steps ((2)-(6)) for σ_0.

8. To generate a^*, do the steps ((2)-(6)) for a.
9. Set $i = i+1$.
10. Repeat steps ((3)-(9)), N times.
11. Obtain the BEs of γ, σ_0 and a using MCMC under SE loss function as

$$\tilde{\theta}_{SE} = \frac{1}{N-M}\sum_{i=M+1}^{N}\theta^{(i)}, \quad \text{where } \theta \text{ is } \gamma, \sigma_0 \text{ or } a.$$

12. Obtain the BEs of γ, σ_0 and a using MCMC under LINEX loss function as

$$\tilde{\theta}_{LINEX} = -\frac{1}{c}\log\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\,\theta^{(i)}}\right], \quad \text{where } \theta \text{ is } \gamma, \sigma_0 \text{ or } a.$$

5. Application

In this section, a real dataset is used to illustrate the proposed procedure in Sections 3 and 4. The censored data in Table 5.1 from Nelson (1990) (page 158) represents the failure time in hours of 30 motors with class-B insulation run at 150°C, 170°C and 190°C. For each level of temperature, ten motors were periodically examined for insulation failure, and the given failure time is the midway between the inspection time when the failure was found and the time of the previous inspection. The test purpose was to estimate the median life of such insulation at its design temperature of 130°C. The + in Table 5.1 indicates a running motor at that number of hours.

Table 5.1: The failure times in hours of 30 motors

$150\ ^{\circ}$C	$170\ ^{\circ}$C	$190\ ^{\circ}$C
9429+	1764	408
9429+	2772	408
9429+	3444	1344
9429+	3542	1344
9429+	3780	1440
9429+	4860	1920
9429+	5196	2256
9429+	6206	2352
9429+	6792+	2596
9429+	6792+	3120+

Based on engineering experience, the Arrhenius relationship is expected to be adequate to describe the accelerated temperature. Thus, the acceleration model can be represented as

$$\ln(\sigma_k) = \alpha_0 + \frac{\alpha_1}{-\phi_k}, \qquad \alpha_1 > 0, k = 0,1,2,3.$$

In this example, $\phi_0 = 130^{\circ}$C, $\phi_1 = 150^{\circ}$C, $\phi_2 = 170^{\circ}$C and $\phi_3 = 190^{\circ}$C.

The data in Table 5.1 is progressively censored, so the progressive censoring schemes $R_{k,i}$, $k = 2,3$, $i = 1,\ldots,m_k$ of each stress level are as follows:

- Under $\phi_2 = 170^o C$: $n_2 = 10$, $m_2 = 8$ and $R_{2,i} = 0$, $i = 1,...,7$, $R_{2,8} = 2$.
- Under $\phi_3 = 190^o C$: $n_3 = 10$, $m_3 = 9$ and $R_{3,i} = 0$, $i = 1,...,8$, $R_{3,9} = 1$.

To check the validity of EE distribution with the data in Table 5.1 for each constant-stress $\phi_k, k = 1,2,3$. We use modified Kolmogorov-Smirnov (K-S) goodness of fit test for progressive type-II censored data. The modified K-S statistic for progressive type-II censored data was suggested by Pakyari and Balakrishnan (2012). The values of the modified K-S statistic and the corresponding P-values under each stress level are presented in Table 5.2. It is clear that the estimated EE distributions provide good fit to the given data due to all P-values are greater than 0.05.

Table 5.2: Test statistic and the corresponding P-values of each stress level for EE distribution

Stress (temperature)	170 ^{o}C	190 ^{o}C
Statistic	0.14637	0.2596
P-value	0.98	0.092

The MLEs and BEs under SE loss function (BSEL) and LINEX loss function (BLL) of the parameters γ, σ_0 and a under progressive censoring schemes $R_{k,i}$ are introduced in Table 5.3. Furthermore, the mean time to failure (MTTF) at different levels of temperature are computed by using MLEs of the parameters and introduced in Table 5.4, where MTTF under normal operating conditions is given by

$$MTTF = \frac{e^1}{\hat{\sigma}_0 \hat{\gamma}} \Gamma(\frac{1}{\hat{\gamma}}, 1),$$

where $\Gamma(.,.)$ is the incomplete gamma function.

Table 5.3: MLEs and BEs of γ, σ_0 and a for the real dataset

$\hat{\theta}$	MLEs	BSEL	BLL		
			$c = -2$	$c = .001$	$c = 2$
$\hat{\gamma}$	0.42420	1.4062	2.2059	1.4061	1.0979
$\hat{\sigma}_0$	0.00025	0.000014	0.000016	0.000014	0.000011
$\hat{a}$	1.9273	7.4753	30.899	7.4664	2.9139

Table 5.4: Mean time to failure at different levels of temperature

temperature	130 ^{o}C	170 ^{o}C	190 ^{o}C
MTTF	25085.8	3945.5	1563.11

6. Interval estimation

In this section, the approximate, bootstrap and credible CIs of the parameters γ, σ_0 and a are derived.

6.1. Approximate confidence intervals

In this subsection, the approximate CIs of the parameters are derived based on the asymptotic distribution of the MLEs of the elements of the vector of unknown parameters $\Theta = (\gamma, \sigma_0, a)$. It is known that the asymptotic distribution of the MLEs of Θ is given by Miller (1981).

$$\left((\hat{\gamma}-\gamma),(\hat{\sigma}_0-\sigma_0),(\hat{a}-a)\right) \rightarrow N\left(0, \sigma_{ij}\right),$$

where $\sigma_{ij}, i, j = 1,2,3$ is the variance-covariance matrix of the unknown parameters γ, σ_0 and a.

The approximate $100\,(1-\alpha)\%$ two sided CI of θ is given by

$$\left(\hat{\theta}_l, \hat{\theta}_u\right) = \hat{\theta} \pm Z_{1-\alpha/2}\sqrt{\sigma_{ii}}, \quad i = 1,2 \; or \; 3, \tag{6.1}$$

where θ is γ, σ_0 or a, and Z_q is the $100q-th$ percentile of a standard normal distribution.

6.2 Bootstrap confidence intervals

CIs based on the parametric bootstrap method for the unknown parameters γ, σ_0 and a using percentile interval are constructed, for more details see Efron and Tibshirani (1993).

The following algorithm is implemented to obtain a bootstrap samples.

Algorithm (2)

1. From an original data, $t_k \equiv \left(t_{k1}, t_{k2}, ..., t_{km_k}\right),\ k = 1,2,...,s,$ compute the MLEs of the parameters γ, σ and a.
2. Use $\hat{\gamma}_{MLE}$, $\hat{\sigma}_{0MLE}$ and $\hat{a}_{MLE}$ to generate a bootstrap sample t_k^* with same $R_{k,i}$, $i = 1,2,...,m_k, k = 1,2,...,s.$
3. As in step (1) based on t_k^* compute the bootstrap samples estimates $\hat{\gamma}^*$, $\hat{\sigma}_0{}^*$ and $\hat{a}^*$ of $\hat{\gamma}_{MLE}$, $\hat{\sigma}_{0MLE}$ and $\hat{a}_{MLE}$ respectively.
4. Repeat steps ((1)-(3)), B times and arrange each estimation in ascending order to obtain the bootstrap samples $\{\hat{\gamma}^{*[1]}, \hat{\gamma}^{*[2]}, ..., \hat{\gamma}^{*[B]}\}$, $\{\hat{\sigma}_0{}^{*[1]}, \hat{\sigma}_0{}^{*[2]}, ..., \hat{\sigma}_0{}^{*[B]}\}$ and $\{\hat{a}^{*[1]}, \hat{a}^{*[2]}, ..., \hat{a}^{*[B]}\}$. Then, the $100\,(1-\alpha)\%$ percentile bootstrap CI of θ is given by

$$\left(\hat{\theta}_l^*, \hat{\theta}_u^*\right) = \left(\hat{\theta}^{*[\alpha B/2]}, \hat{\theta}^{*[(1-\alpha/2)B]}\right), \quad \text{where } \theta \text{ is } \gamma, \sigma_0 \text{ or } a. \tag{6.2}$$

6.3 Credible confidence intervals

A $100(1-\alpha)\%$ Bayesian credible or posterior interval of a random quantity θ is the interval that has posterior probability $(1-\alpha)$, that is

$$p(l \le \theta \le u) = \int_l^u \pi_1^*(\theta \mid t)\, d\theta = 1-\alpha.$$

There are different types of credible intervals, including a central interval of posterior probability which is the range of values between the $\alpha/2$ and $(1-\alpha)/2$ percentiles.
The following algorithm is performed to obtain credible CIs of γ, σ_0 and a.

Algorithm (3)

1. Do steps $((1)-(11))$ in algorithm (1).
2. Repeat step (1), K times and arrange each estimation in ascending order as $\{\tilde{\gamma}^{[1]},\tilde{\gamma}^{[2]},...,\tilde{\gamma}^{[K]}\}$, $\{\tilde{\sigma}_0^{[1]},\tilde{\sigma}_0^{[2]},...,\tilde{\sigma}_0^{[K]}\}$ and $\{\tilde{a}^{[1]},\tilde{a}^{[2]},...,\tilde{a}^{[K]}\}$.
   Then, the $100\,(1-\alpha)\%$ credible CI of θ is given by
   $$\left(\tilde{\theta}_l,\tilde{\theta}_u\right)=\left(\tilde{\theta}^{[\alpha K/2]},\tilde{\theta}^{[(1-\alpha/2)K]}\right),\quad \text{where } \theta \text{ is } \gamma, \sigma_0 \text{ or } a. \tag{6.3}$$

7. Simulation studies

In this section, simulation studies are conducted to investigate the performances of the MLEs and BEs under SE loss function (BSEL) and LINEX loss function (BLL) regarding their mean square errors (MSEs) and relative absolute biases (RABs) for different choices of n_k, m_k and $R_{k,i}$, $i=1,2,...,m_k$, $k=1,2,...,4$. Furthermore, the 95% approximate, credible and percentile bootstrap CIs are computed. The progressive censoring schemes used in the simulation studies are shown in Table 7.1. Moreover, Tables 7.2 and 7.3 introduce MSEs and RABs of the MLEs and BEs of the model parameters. Finally, Tables 7.4 and 7.5 include the lengths and the coverage probabilities of the 95% approximate, credible and percentile bootstrap CIs of the model parameters.

Table 7.1: The progressive censoring schemes $R_{k,i}$, $i=1,2,...,m_k$, $k=1,2,...,4$, used in the simulation studies.

n_k	m_k	$C.S$	$(R_{k,1},...,R_{k,m_k})$	$C.S$	$(R_{k,1},...,R_{k,m_k})$	$C.S$	$(R_{k,1},...,R_{k,m_k})$
$n_k=\begin{cases}25 & k=1\\15 & k=2\\10 & k=3\\5 & k=4\end{cases}$	$m_k=\begin{cases}20 & k=1\\12 & k=2\\8 & k=3\\4 & k=4\end{cases}$	[1]	$R_{k,i}=\begin{cases}5 & k=1,i=1\\3 & k=2,i=1\\2 & k=3,i=1\\1 & k=4,i=1\\0 & other\ wise\end{cases}$	[2]	$R_{k,i}=\begin{cases}5 & k=1,i=m_1\\3 & k=2,i=m_2\\2 & k=3,i=m_3\\1 & k=4,i=m_4\\0 & other\ wise\end{cases}$	[3]	$R_{k,i}=\begin{cases}1 & k=1,i=12,...,16\\1 & k=2,i=7,8,9\\1 & k=3,i=5,6\\1 & k=4,i=3\\0 & other\ wise\end{cases}$
$n_k=\begin{cases}35 & k=1\\25 & k=2\\20 & k=3\\10 & k=4\end{cases}$	$m_k=\begin{cases}27 & k=1\\20 & k=2\\16 & k=3\\8 & k=4\end{cases}$	[4]	$R_{k,i}=\begin{cases}8 & k=1,i=1\\5 & k=2,i=1\\4 & k=3,i=1\\2 & k=4,i=1\\0 & other\ wise\end{cases}$	[5]	$R_{k,i}=\begin{cases}8 & k=1,i=m_1\\5 & k=2,i=m_2\\4 & k=3,i=m_3\\2 & k=4,i=m_4\\0 & other\ wise\end{cases}$	[6]	$R_{k,i}=\begin{cases}1 & k=1,i=13,...,20\\1 & k=2,i=11,...,15\\1 & k=3,i=9,...,12\\1 & k=4,i=5,6\\0 & other\ wise\end{cases}$
$n_k=\begin{cases}50 & k=1\\35 & k=2\\25 & k=3\\20 & k=4\end{cases}$	$m_k=\begin{cases}38 & k=1\\27 & k=2\\20 & k=3\\16 & k=4\end{cases}$	[7]	$R_{k,i}=\begin{cases}12 & k=1,i=1\\8 & k=2,i=1\\5 & k=3,i=1\\4 & k=4,i=1\\0 & other\ wise\end{cases}$	[8]	$R_{k,i}=\begin{cases}12 & k=1,i=m_1\\8 & k=2,i=m_2\\5 & k=3,i=m_3\\4 & k=4,i=m_4\\0 & other\ wise\end{cases}$	[9]	$R_{k,i}=\begin{cases}1 & k=1,i=18,...,29\\1 & k=2,i=12,...,19\\1 & k=3,i=10,...,14\\1 & k=4,i=9,...,12\\0 & other\ wise\end{cases}$

$n_k = \begin{cases} 70 & k=1 \\ 50 & k=2 \\ 35 & k=3 \\ 30 & k=4 \end{cases}$	$m_k = \begin{cases} 55 & k=1 \\ 40 & k=2 \\ 27 & k=3 \\ 24 & k=4 \end{cases}$	[10]	$R_{k,i} = \begin{cases} 15 & k=1, i=1 \\ 10 & k=2, i=1 \\ 8 & k=3, i=1 \\ 6 & k=4, i=1 \\ 0 & other\ wise \end{cases}$	[11]	$R_{k,i} = \begin{cases} 15 & k=1, i=m_1 \\ 10 & k=2, i=m_2 \\ 8 & k=3, i=m_3 \\ 6 & k=4, i=m_4 \\ 0 & other\ wise \end{cases}$	[12]	$R_{k,i} = \begin{cases} 1 & k=1, i=34,...,48 \\ 1 & k=2, i=26,...,35 \\ 1 & k=3, i=15,...,22 \\ 1 & k=4, i=14,...,19 \\ 0 & other\ wise \end{cases}$

The estimation procedure is performed according to the following algorithm.

Algorithm (4)

1. Specify the values of s, n_k, m_k, ϕ_0, ϕ_1,..., ϕ_s and c.
2. For given values of the prior parameters μ, λ and β generate γ from $Gamma(\mu, \lambda)$ and σ_0 from $Exponential(\beta)$.
3. Generate s simple random samples of size m_k from Uniform$(0,1)$ distribution, $(U_{k1}, U_{k2}, ..., U_{km_k})$, $k = 1,2,...,s$.
4. Determine the values of the censored schemes, $R_{k,i}$, $i = 1,2,...,m_k$, and $k = 1,2,...,s$ such that $\sum_{i=1}^{m_k} R_{k,i} = n_k - m_k$.
5. Set $E_{ki} = U_{ki}^{1/(i+\sum_{d=m_k-i+1}^{m_k} R_{k,d})}$, $i = 1, 2, ..., m_k$, and $k = 1,2,...,s$.
6. Obtain the progressive type-II censored samples $(U_{k1}^*, U_{k2}^*, ..., U_{km_k}^*)$, where $U_{ki}^* = 1 - \prod_{d=m_k-i+1}^{m_k} E_{kd}$, $i = 1,2,...,m_k$, $k = 1,2,...,s$.
7. Use step 6, to generate random samples $(t_{k1}, ..., t_{km_k})$, $k = 1,2,...,s,$ from (2.6), as follows:
$$t_{ki} = \frac{1}{a^k \sigma_0}\left[\left(1 - \log(1 - U_{ki}^*)\right)^{\frac{1}{\gamma}} - 1\right],\ i = 1,2,...,m_k, k = 1,2,...,s.$$
8. Use the progressive censored data to compute the MLEs of the model parameters by solving the nonlinear system ((3.4)-(3.6)).
9. Compute the BEs of the model parameters relative to SE and LINEX loss functions using the algorithm (1), with $N = 11000$, $M = 1000$.
10. Compute the approximate CIs with confidence level 95% for the three parameters γ, σ_0 and a.
11. Compute 95% bootstrap and credible CIs using the algorithm (2) and algorithm (3) respectively of the parameters γ, σ_0 and a.
12. Replicate the steps $((3)-(11))$, 1000 times.
13. Compute the average values of the MSEs and RABs associated with the MLEs and BEs of the parameters γ, σ_0 and a.

14. Do steps ((1)-(13)) with different values of n_k, m_k and $R_{k,i}$, $i=1,2,...,m_k$, $k=1,2,...,s$.

Table 7.2: MSEs and RABs inside the parentheses for MLEs and BEs under SE loss function (BSEL) and LINEX loss function (BLL) of γ, σ_0 and a with true values ($\gamma=0.9181, \sigma_0=1.0108$, $a=3.2102$, $\mu=8.429$, $\lambda=0.1089$ and $\beta=1.0108$), the number of stress levels $(s=4)$, and $\phi_0=50$, $\phi_1=80$, $\phi_2=110$, $\phi_3=140$ and $\phi_4=170$.

$\sum_{k=1}^{s} n_k$	$\sum_{k=1}^{s} m_k$	$C.S$	θ	ML	BSEL	BLL		
						$c=-2$	$c=.001$	$c=2$
	44	[1]	γ	1.1437(0.6857)	0.1077(0.2638)	0.3435(0.4711)	0.1077(0.2638)	0.0495(0.1815)
			σ_0	0.6099(0.5681)	0.2510(0.3626)	1.9585(1.1007)	0.2510(0.3625)	0.1717(0.2617)
			a	0.2364(0.1204)	0.1311(0.0933)	0.23781(0.1166)	0.1311(0.0933)	0.1462(0.0994)
		[2]	γ	1.7308(0.8929)	0.1008(0.2635)	0.5873(0.6304)	0.1007(0.2634)	0.0386(0.1691)
			σ_0	0.4385(0.5380)	0.2466(0.3831)	1.9363(1.0805)	0.2465(0.3830)	0.1619(0.2680)
			a	0.2065(0.1166)	0.1408(0.1002)	0.2532(0.1216)	0.1408(0.10029)	0.1747(0.1101)
		[3]	γ	0.6778(0.4794)	0.0625(0.1962)	0.2134(0.3555)	0.0624(0.1962)	0.0398(0.1626)
			σ_0	0.4648(0.5453)	0.2270(0.3768)	2.8348(1.3936)	0.2268(0.3767)	0.1925(0.2686)
			a	0.2674(0.1284)	0.1647(0.1045)	0.2402(0.1919)	0.1648(0.1045)	0.2338(0.1300)
	55		γ	0.3463(0.3340)	0.0607(0.1877)	0.1723(0.3039)	0.0607(0.1877)	0.0331(0.1462)
			σ_0	0.4195(0.5019)	0.2260(0.3516)	1.8804(1.0476)	0.2258(0.3515)	0.1253(0.2451)
			a	0.2028(0.1112)	0.1219(0.0962)	0.1859(0.1069)	0.1219(0.0962)	0.1970(0.1131)
	71	[4]	γ	0.1508(0.2565)	0.0508(0.1801)	0.0775(0.2116)	0.0508(0.1801)	0.0375(0.1485)
			σ_0	0.5112(0.4936)	0.2403(0.3585)	1.8242(0.9467)	0.2401(0.3584)	0.1018(0.2814)
			a	0.1635(0.0963)	0.1172(0.0809)	0.1816(0.1022)	0.1172(0.0808)	0.1096(0.0845)
		[5]	γ	0.6015(0.4754)	0.1164(0.2663)	0.2902(0.4831)	0.1164(0.2663)	0.0382(0.1685)
			σ_0	0.4379(0.5364)	0.2133(0.3777)	1.9351(1.0312)	0.2132(0.3776)	0.0790(0.2723)
			a	0.1574(0.10813)	0.1089(0.0856)	0.1643(0.1618)	0.1089(0.0856)	0.17389(0.0927)
		[6]	γ	0.7201(0.4288)	0.0734(0.2099)	0.1730(0.3271)	0.0734(0.2098)	0.0392(0.1548)
			σ_0	0.3860(0.4687)	0.0912(0.2441)	0.5477(0.5337)	0.0911(0.2441)	0.0726(0.2595)
			a	0.2236(0.1116)	0.1523(0.0926)	0.2217(0.1128)	0.1522(0.0926)	0.1327(0.0892)
	90		γ	0.1050(0.2415)	0.0464(0.1705)	0.0917(0.2309)	0.0464(0.1705)	0.0317(0.1448)
			σ_0	0.3850(0.4131)	0.2166(0.3367)	1.2895(0.8471)	0.2165(0.3366)	0.0823(0.2445)
			a	0.1195(0.0881)	0.0976(0.0801)	0.1101(0.0787)	0.0976(0.0801)	0.1101(0.0882)
	101	[7]	γ	0.1016(0.2529)	0.3908(0.1769)	0.0964(0.2604)	0.0398(0.1769)	0.0303(0.1464)
			σ_0	0.1877(0.3517)	0.1228(0.2757)	0.5116(0.4854)	0.1228(0.2757)	0.0878(0.2437)
			a	0.0384(0.0475)	0.0359(0.0466)	0.0325(0.0443)	0.0359(0.0466)	0.0516(0.0588)
		[8]	γ	0.8670(0.5685)	0.0750(0.2189)	0.2883(0.4263)	0.0749(0.2189)	0.0598(0.1992)
			σ_0	0.2625(0.4164)	0.1060(0.2458)	0.7101(0.5208)	0.1059(0.2458)	0.0765(0.2275)
			a	0.1215(0.0832)	0.0422(0.0506)	0.1230(0.0818)	0.0422(0.0506)	0.1009(0.0836)
		[9]	γ	0.1318(0.2679)	0.0601(0.194)	0.1137(0.2621)	0.0600(0.1942)	0.0385(0.1616)
			σ_0	0.1374(0.3092)	0.0946(0.2394)	0.2733(0.3743)	0.0946(0.2394)	0.0669(0.2213)

			a	0.0666(0.0668)	0.0521(0.0589)	0.0791(0.0738)	0.0521(0.0589)	0.0457(0.0520)

Table 7.3 *(Continued)*

$\sum_{k=1}^{s} n_k$	$\sum_{k=1}^{s} m_k$	$C.S$	θ	ML	BSEL	BLL		
						$c=-2$	$c=.001$	$c=2$
	130		γ	0.0645(0.2101)	0.0391(0.1690)	0.0677(0.2167)	0.0391(0.1690)	0.0253(0.1398)
			σ_0	0.1490(0.3098)	0.0921(0.2224)	0.2051(0.2945)	0.0921(0.2224)	0.0774(0.2289)
			a	0.0798(0.0611)	0.0675(0.0567)	0.0812(0.0626)	0.0675(0.0567)	0.0648(0.0595)
	146	[10]	γ	0.0528(0.1802)	0.0407(0.1603)	0.0619(0.1871)	0.0407(0.1603)	0.0296(0.1455)
			σ_0	0.1122(0.2734)	0.0968(0.2587)	0.2520(0.3883)	0.0968(0.2587)	0.0641(0.2087)
			a	0.0470(0.0566)	0.0332(0.0446)	0.0478(0.0562)	0.0332(0.0446)	0.0465(0.0561)
		[11]	γ	0.7572(0.4231)	0.0728(0.2114)	0.2954(0.3779)	0.0727(0.2113)	0.0358(0.1498)
			σ_0	0.1692(0.3334)	0.0930(0.2537)	0.3688(0.4301)	0.0930(0.2537)	0.0738(0.2271)
			a	0.0704(0.0706)	0.0609(0.0656)	0.0710(0.0701)	0.0609(0.0655)	0.0588(0.063)
		[12]	γ	0.0601(0.1772)	0.0340(0.1459)	0.0547(0.1887)	0.0340(0.1459)	0.0231(0.1193)
			σ_0	0.1376(0.2713)	0.0957(0.2203)	0.2479(0.3555)	0.0956(0.2203)	0.0663(0.1954)
			a	0.0605(0.0592)	0.0513(0.0538)	0.0573(0.0588)	0.0513(0.0538)	0.0531(0.0541)
	185		γ	0.0294(0.1422)	0.0238(0.1330)	0.0345(0.1561)	0.0238(0.1330)	0.0177(0.1155)
			σ_0	0.1031(0.2565)	0.0880(0.2304)	0.1880(0.3054)	0.0880(0.2304)	0.0649(0.2086)
			a	0.0597(0.0647)	0.0550(0.0516)	0.0594(0.0605)	0.0555(0.0516)	0.0565(0.0582)

Table 7.4 Lengths and coverage probabilities of the 95% approximate, credible and bootstrap CIs for γ, σ_0 and a with true values ($\gamma=0.9181, \sigma_0=1.0108$, $a=3.2102$, $\mu=8.429$, $\lambda=0.1089$ and $\beta=1.0108$), the number of stress levels $(s=4)$, and $\phi_0=50$, $\phi_1=80$, $\phi_2=110$, $\phi_3=140$ and $\phi_4=170$.

$\sum_{k=1}^{s} n_k$	$\sum_{k=1}^{s} m_k$	$C.S$	θ	Length			Coverage probability		
				Approximate CI	Credible CI	Bootstrap CI	Approximate CI	Credible CI	Bootstrap CI
	44	[1]	γ	3.609	1.353	4.037	1	1	0.75
			σ_0	2.949	2.017	4.734	0.725	0.95	0.725
			a	1.944	1.819	2.708	0.95	1	0.975
		[2]	γ	5.399	1.615	5.146	1	1	0.85
			σ_0	3.000	2.357	3.735	0.8	1	0.825
			a	1.896	1.807	2.575	0.975	0.975	0.95
		[3]	γ	3.162	1.233	4.262	0.975	1	0.85
			σ_0	3.238	2.482	6.864	0.925	1	0.85
			a	1.851	1.718	2.612	0.975	1	0.95
	55		γ	1.922	1.088	3.033	0.975	1	0.95
			σ_0	2.678	2.242	3.423	0.95	0.975	0.875
			a	1.757	1.625	2.374	0.925	0.975	0.925
	71	[4]	γ	1.262	0.921	1.891	0.9	1	0.85

			σ_0	2.481	2.085	4.242	0.9	0.975	0.825
			a	1.582	1.482	2.418	0.9	0.975	0.925

Table 7.3 *(Continued)*

$\sum_{k=1}^{s} n_k$	$\sum_{k=1}^{s} m_k$	$C.S$	θ	Length			Coverage probability		
				Approximate CI	Credible CI	Bootstrap CI	Approximate CI	Credible CI	Bootstrap CI
		[5]	γ	2.869	1.312	3.488	0.975	0.975	0.925
			σ_0	2.990	2.227	3.316	0.925	1	0.875
			a	1.501	1.4369	1.544	1	1	1
		[6]	γ	2.291	1.117	3.126	1	0.975	0.9
			σ_0	2.542	1.754	2.950	0.95	1	0.9
			a	1.511	1.4401	1.700	0.9	0.95	0.925
	90		γ	1.067	0.875	1.683	0.95	0.95	0.9
			σ_0	2.231	1.918	3.281	0.9	0.975	0.875
			a	1.347	1.281	1.450	0.95	0.95	0.9
	101	[7]	γ	1.1967	0.935	1.831	0.975	1	0.925
			σ_0	1.639	1.572	2.279	0.875	0.975	0.875
			a	1.132	1.106	1.255	1	1	0.975
		[8]	γ	3.235	1.292	3.484	0.95	0.975	0.825
			σ_0	2.009	1.702	2.166	0.875	0.975	0.85
			a	1.155	1.128	1.210	0.975	0.95	0.95
		[9]	γ	1.235	0.940	2.016	0.975	0.95	0.85
			σ_0	1.565	1.490	2.237	0.875	1	0.825
			a	1.207	1.171	1.348	1	0.975	0.975
	130		γ	0.962	0.805	1.419	1	0.975	0.875
			σ_0	1.385	1.347	1.862	0.975	0.95	0.9
			a	1.034	1.018	1.144	0.95	0.95	0.95
	146	[10]	γ	0.798	0.707	1.123	0.975	0.925	0.875
			σ_0	1.509	1.402	1.658	0.95	1	0.925
			a	0.983	0.955	1.108	0.975	0.975	0.925
		[11]	γ	2.186	1.177	2.601	0.975	0.975	0.875
			σ_0	1.716	1.537	1.782	0.875	0.975	0.85
			a	0.955	0.946	0.949	0.95	0.95	250.875
		[12]	γ	0.951	0.808	1.295	0.975	0.95	0.95
			σ_0	1.481	1.415	1.889	0.9	0.9	0.9
			a	0.982	0.963	1.021	0.95	0.95	0.9
	185		γ	0.679	0.633	0.880	0.925	0.95	0.95
			σ_0	1.255	1.227	1.506	0.95	0.975	0.9
			a	0.870	0.854	0.945	0.95	0.925	0.925

8. Conclusions

In this paper, we have considered the GP as a constant-stress ALT model for the EE distribution under progressive type-II censored data. Based on simulation studies, point estimation of the model parameters γ, σ_0 and a has been investigated through maximum likelihood and Bayes methods. Moreover, approximate, credible and bootstrap CIs have been established for the model parameters γ, σ_0 and a. The calculations have been worked out based on different sample sizes and three different progressive censoring schemes, one of them represents the traditional type-II censoring.

From the results in Tables 7.2, 7.3, 7.4 and 7.5, we observed the following:

1. The MSEs and RABs of MLEs and BEs of the considered parameters decrease as the sample size increases, except for few cases. This exception may be due to fluctuation in data.
2. The BEs of γ, σ_0 and a under SE loss function and LINEX loss function $(c = 0.01, c = 2)$ give more accurate results through the MSEs and RABs than MLEs.
3. The BEs of γ and σ_0 under LINEX loss function $(c = 2)$ have the smallest MSEs and RABs as compared with BEs under SE loss function and MLEs, except for few cases.
4. The BEs of the parameter a under SE loss function have the smallest MSEs and RABs as compared with BEs under LINEX loss function and MLEs, except for few cases.
5. The length of the approximate, bootstrap and credible CIs decreases as the sample size increases.
6. The credible CIs of γ, σ_0 and a give more accurate results than approximate and bootstrap CIs through the length and the coverage probability of CIs.

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