On Generating a New Family of Distributions Using the Tangent Function

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Abstract

In this paper, a method for generating a new family of univariate continuous distributions using the tangent function is proposed. Some general properties of this new family are discussed: hazard function, *quantile function*, *Rényi* and *Shannon entropies*, symmetry, and existence of the non-central n^{th} moment. Some new members as sub-families in the $T-X$ family of distributions are provided. Three members of the new sub-families are defined and discussed: the four-parameter Normal-Generalized hyperbolic secant distribution ($NGHS$), the four-parameter Gumbel-Generalized hyperbolic secant distribution ($GGHS$), and the five-parameter Generalized Error-Generalized hyperbolic secant distribution ($GEHS$), the shapes of these distributions were found: skewed right, skewed left, or symmetric, and unimodal, bimodal, or trimodal. Finally, to demonstrate the usefulness and the capability of the distributions, two real data sets are used and the results are compared with other known distributions.

Keywords: Generalized class; Hazard function; Moments; Quantiles; Shannon entropy.

1. Introduction

Statistical distribution is a mathematical description of a random phenomenon in terms of the probabilities of events. Many methods were recently proposed and developed to generate new statistical distributions.

A class of beta-generated distributions were proposed and studied by Eugene, Lee and Famoye (Eugene et al., 2002). Their idea has been build depending on the property of the beta random variable lies in the interval $(0,1)$, and they have defined the cumulative distribution function (CDF) of the beta-generated class by

$$G(x) = \int_0^{F(x)} b(t)dt\,, \alpha, \beta > 0 \tag{1.1}$$

where $b(t) = \left(B(\alpha,\beta)\right)^{-1} t^{\alpha-1}(1-t)^{\beta-1}$ is the beta distribution and $F(x)$ is the CDF of any continuous random variable X. The probability density function (PDF) corresponding to this class in (1.1) is given by

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)\left(F(x)\right)^{\alpha-1}\left(1-F(x)\right)^{\beta-1} \tag{1.2}$$

Many studies about the beta-generated class have been published by applying different F in (1.2). Examples include: Beta-Gumbel distribution by Nadarajah and Kotz (2004), beta-exponential distribution by Nadarajah and Kotz (2006), Beta-Weibull distribution by Famoye et al. (2004), beta-gamma distribution by Kong et al. (2007), Beta-Pareto distribution by Akinsete et al. (2008), beta-modified Weibull distribution by Silva et al.

(2010), Beta-Birnbaum-Saunders distribution by Cordeiro and Lemonte (2011), and Beta-Cauchy distribution by Alshawarbeh et al. (2012).

The beta-generated class was extended by Jones (2009) and Cordeiro and de Castro (2011), that was by replacing the beta distribution $b(t)$ in (1.1) by the Kumaraswamy distribution $b(t) = \alpha\beta t^{\alpha-1}(1-t^{\alpha})^{\beta-1}, t \in (0,1)$ (Kumaraswamy, 1980). The new Kumaraswamy-generated (Kw-G) class is given by

$$g(x) = \alpha\beta f(x)\left(F(x)\right)^{\alpha-1}\left(1-\left(F(x)\right)^{\alpha}\right)^{\beta-1} \tag{1.3}$$

As is the beta-generated class, several distributions and generalized distributions from (1.3) have been studied: Kw-Weibull distribution by Cordeiro et al. (2010), Kw-Gumbel distribution by Cordeiro et al. (2011), Kw-generalized gamma distribution by de Castro et al. (2011), and the Kw-generalized half-normal distribution by Cordeiro et al. (2012).

By replacing the beta distribution $b(t)$ in (1.1) by the generalized beta distribution of the first kind $b(t) = c\left(B(\alpha,\beta)\right)^{-1} t^{\alpha c-1}(1-t^{c})^{\beta-1}, t \in (0,1)$ (McDonald, 1984) we get the generalized beta-generated (GB-G) class, this family of distributions were proposed by Alexander et al. (2012), and it is given by

$$g(x) = \frac{1}{B(\alpha,\beta)} c f(x)\left(F(x)\right)^{\alpha c-1}\left(1-\left(F(x)\right)^{c}\right)^{\beta-1} \tag{1.4}$$

A more general class from classes in (1.2), (1.3) and (1.4) has been introduced by Alzaatreh et al. (2013a). This new class is depending on replacing the beta PDF $b(t)$ with a PDF $r(t)$ of a continuous random variable $T \in [a,b]$, $-\infty \le a < b \le \infty$, and the CDF $F(x)$ with a function $W(F(x))$, where $W(\cdot)$ satisfies the following conditions:

i. $W(F(x)) \in [a,b]$.
ii. W is differentiable and monotonically non-decreasing.
iii. $W(F(x)) \longrightarrow a$ as $x \longrightarrow -\infty$, and $W(F(x)) \longrightarrow b$ as $x \longrightarrow +\infty$.

(1.5)

The CDF of this family of distributions is called $T - X(W)$ family and is defined as

$$G(x) = \int_{a}^{W(F(x))} r(t)dt = R\left\{W\left(F(x)\right)\right\} \tag{1.6}$$

The corresponding PDF (if it exists) of this class is given by

$$g(x) = \frac{d}{dx}G(x) = \left\{\frac{d}{dx} W\left(F(x)\right)\right\} r\left\{W\left(F(x)\right)\right\} \tag{1.7}$$

where $R(\cdot)$ is the CDF of a continuous random variable T. Alzaatreh et al. (2013a) have suggested some $W(\cdot)$ functions. As a special case of $T - X(W)$ family defined in (1.6), Alzaatreh et al. (2013a) considered the function $W(F(x)) = -\log\{1-F(x)\}$ with $(0,\infty)$ as the support of the random variable T. Whereas Al-Aqtash et al. (2015) used the $W(\cdot)$ as a logit function of the CDF $F(x)$, $W(F(x)) = \log(F(x)/(1-F(x)))$ with $(-\infty,\infty)$ as the support of the random variable T.

Aljarrah el al. (2014) have generalized the $T-X(W)$ family were proposed by Alzaatreh et al. (2013a), this new generalized family of distributions is called $T-X\{Y\}$ family. This family is depending on the quantile of the random variable Y, Q_Y, where the random variable Y has the CDF $P(y)$. They defined $W(\lambda)=Q_Y(\lambda)$ where $0<\lambda<1$ which satisfies the conditions in (1.5). The CDF of the new family is defined as

$$G(x)=\int_a^{Q_Y(F(x))} r(t)dt = R\{Q_Y(F(x))\} \tag{1.8}$$

and the corresponding PDF (if it exists) is defined as

$$g(x)=\frac{d}{dx}G(x)=\frac{f(x)}{p\{Q_Y(F(x))\}}r\{Q_Y(F(x))\} \tag{1.9}$$

In this paper, a new $W(\cdot)$ function is used; this function is depending on the tangent function with $(-\infty,\infty)$ as the support of the random variable T. In Section 2, the family of $T-X$ distributions depending on the tangent function is defined and some of its general properties are discussed: hazard function, *quantile function, Rényi* and *Shannon entropies*, symmetry, and existence of the non-central n^{th} moment. In Section 3, will define and discuss some new members as sub-families in the $T-X$ family of distributions with different T distributions: Normal$-X$ sub-family, Cauchy$-X$ sub-family, and Generalized Error$-X$ sub-family, also three members of the new sub-families are defined and discussed: the four-parameter Normal-Generalized hyperbolic secant distribution ($NGHS$), the four-parameter Gumbel-Generalized hyperbolic secant distribution ($GGHS$), and the five-parameter Generalized Error-Generalized hyperbolic secant distribution ($GEHS$). In Section 4, two real data sets are used to demonstrate the usefulness of this new family of distributions. In Section 5, the summary and conclusion. The programs were used to compute the results in sections 4.1 and 4.2 were written in R 3.3.1 programming language (R Core Team, 2016). These R codes are available to the reader from the author.

2. Generating new family of distributions using the tangent function

In this section, a new class of distributions will be proposed. Let T be a continuous random variable with PDF $r(t)$ and CDF $R(t)$, defined on $(-\infty,\infty)$, and let X be any continuous random variable with PDF $f(x)$ and CDF $F(x)$.

Defined $W(F(x))=a+b\tan(\pi(F(x)-1/2))$, where $a\in\mathbb{R}$ and $b\in(0,\infty)$, note that W is satisfied the conditions in (1.5):

i. Since $\pi(F(x)-1/2)\in[-\pi/2,\pi/2]$, then $t=a+b\tan(\pi(F(x)-1/2))\in(-\infty,\infty)$.
ii. The tangent function is differentiable and monotonically non-decreasing on $[-\pi/2,\pi/2]$.
iii. Since $F(x)\to 0$ as $x\to-\infty$, then $\pi(F(x)-1/2)\to-\pi/2$ and $a+b\tan(\pi(F(x)-1/2))\to-\infty$. Similarly, $F(x)\to 1$ as $x\to\infty$, then $\pi(F(x)-1/2)\to\pi/2$ and $a+b\tan(\pi(F(x)-1/2))\to\infty$.

The CDF of the new $T-X$ family of distributions is defined by

$$G(x)=\int_{-\infty}^{a+b\tan\left(\pi(F(x)-1/2)\right)} r(t)dt = R\left\{a+b\tan\left(\pi\left(F(x)-\frac{1}{2}\right)\right)\right\} \qquad (2.1)$$

The corresponding PDF of this family is given by

$$g(x)=\pi b\sec^2\left(\pi\left(F(x)-\frac{1}{2}\right)\right)f(x)\,r\left\{a+b\tan\left(\pi\left(F(x)-\frac{1}{2}\right)\right)\right\} \qquad (2.2)$$

Note: Aljarrah et al (2014) were mentioned about $T-X\{\text{Cauchy}\}$ family in his **Table 1**. The family in (2.1) is the same their family.

Denote X_f is the random variable of X with PDF f, and X_g is the random variable of X with PDF g. Since X_f is any continuous random variable, so it can be easily derived many new $T-X$ family of distributions.

Some remarks on the $T-X$ family of distributions defined in (2.1):

(a) Since for any angle θ, $\sec(\pi/2-\theta)=\csc(\theta)$ and $\tan(\theta-\pi/2)=-\cot(\theta)$, the PDF in (2.2) can be written as

$$g(x)=\pi b\csc^2\left(\pi F(x)\right)f(x)\,r\left\{a-b\cot\left(\pi F(x)\right)\right\} \qquad (2.3)$$

and the CDF in (2.1) can be written as

$$G(x)=R\left\{a-b\cot\left(\pi F(x)\right)\right\} \qquad (2.4)$$

(b) The relation $G(x)=R(a-b\cot(\pi F(x)))=R(t)$ gives the relationship between the random variable X_g with PDF in (2.3) and the random variable T, where $T=a-b\cot(\pi F(X_g))$, and this implies $X_g=F^{-1}\left\{\cot^{-1}\left((a-T)/b\right)/\pi\right\}$. This result it can be used for: Simulating the random variable X_g by first simulating the random variable T from PDF $r(t)$, and then applying the transformation $X_g=F^{-1}\left\{\cot^{-1}\left((a-T)/b\right)/\pi\right\}$. Also, computing the central moments of the random variable $X_g$by $E\left[X_g^n\right]=E\left\{\left[F^{-1}\left\{\cot^{-1}\left((a-T)/b\right)/\pi\right\}\right]^n\right\}$.

(c) The moment generating function of the random variable X_g, $M_{X_g}(s)$, where $-\infty<s<\infty$, is defined as

$$M_{X_g}(s)=E(e^{sX_g})=E\left(e^{\frac{1}{\pi}F^{-1}\left\{\cot^{-1}\left(\frac{a-T}{b}\right)\right\}s}\right)=\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{s}{\pi}\right)^i E\left[\left(F^{-1}\left\{\cot^{-1}\left(\frac{a-T}{b}\right)\right\}\right)^i \qquad (2.5)$$

(d) If $a=0$ and $b=1$, and the random variable T has the standard Cauchy Distribution, then the CDF of the random variable X_g, $G(x)$, in (2.4) reduces to $F(x)$. Similarly, if $a=0$ and $b=1$, and the random variable X_g has the standard Cauchy Distribution, then the CDF of the random variable X_g, $G(x)$, in (2.4) reduces to $R(x)$.

(e) The hazard function, $h_g(x) = g(x)/(1 - G(x))$, for the random variable X_g in (2.4) is defined as

$$h_g(x) = \pi b \csc^2\big(\pi F(x)\big) f(x)\, h_r\{a - b\cot\big(\pi F(x)\big)\} \tag{2.6}$$

where h_r is the hazard function of the random variable T with CDF $R(t)$.

Theorem 1: Let $0 < \lambda < 1$, the *quantile function* of the $T - X$ family of distributions defined in (2.4) is given by

$$Q_{X_g}(\lambda) = Q_{X_f}\left(\frac{1}{\pi}\cot^{-1}\left(\frac{a - Q_T(\lambda)}{b}\right)\right) = F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a - R^{-1}(\lambda)}{b}\right)\right\} \tag{2.7}$$

where$Q_{X_f}(\lambda) = F^{-1}(\lambda)$ is the *quantile function* of the random variable X_f with CDF $F(x)$ and $Q_T(\lambda) = R^{-1}(\lambda)$ is the *quantile function* of the random variable T with CDF $R(t)$.

Proof: By solving $G\left(Q_{X_g}(\lambda)\right) = \lambda$ for x, where $x = Q_{X_g}(\lambda)$, we get $\lambda = G(Q_{X_g}(\lambda)) = R\{a - b\cot(\pi F(Q_{X_g}(\lambda)))\}$, which implies $R^{-1}(\lambda) = a - b\cot(\pi F(Q_{X_g}(\lambda)))$, and so $(a - R^{-1}(\lambda))/b = \cot(\pi F(Q_{X_g}(\lambda)))$.

Now, since $\pi F(Q_{X_g}(\lambda)) \in [0, \pi]$, we can take the inverse of the cotangent function for the both sides, and we obtain $(1/\pi)\cot^{-1}((a - R^{-1}(\lambda))/b) = F(Q_{X_g}(\lambda))$, and $F^{-1}\{(1/\pi)\cot^{-1}((a - R^{-1}(\lambda))/b)\} = Q_{X_g}(\lambda)$. Hence, $Q_{X_f}\big((1/\pi)\cot^{-1}((a - Q_T(\lambda))/b)\big) = Q_{X_g}(\lambda)$ which is the result in (2.7).

Entropy has wide applications in science, engineering and probability theory, for a random variable X_g, the entropy is a measure of the variation of the uncertainty. The *Rényi entropy* (the spectrum of *Rény* information) of order α, of a random variable X_g with PDF $g(x)$, is defined as $H_\alpha(X_g) = -\log\big([H(\alpha)]^{1/(\alpha-1)}\big)$ where $H(\alpha) = \int_{-\infty}^{\infty} g(x) g^{\alpha-1}(x)dx = E\big[g^{\alpha-1}(X_g)\big]$, $\alpha \geq 0$ and $\alpha \neq 1$ (Rényi, 1961).

Let the random variable X_g follow the $T - X$ family of distributions defined in (2.4), then the Rényi entropy of the random variable X_g, $H_\alpha(X_g)$, is given by

$$H_\alpha(X_g) = \frac{1}{1-\alpha}\log\left((\pi b)^\alpha \int_{-\infty}^{\infty} \csc^{2\alpha}\left(\pi\big(F(x)\big)\right) f^\alpha(x) r^\alpha\{a - b\cot\big(\pi F(x)\big)\}dx\right) \tag{2.8}$$

The *Shannon entropy* is defined by Shannon (1948), and it is considered as a special case of the Rényi entropy when $\alpha \to 1$. The *Shannon entropy* of a random variable X with PDF $g(x)$, is defined as $E\big(-\log\big(g(X_g)\big)\big)$.

Theorem 2: Let the random variable X_g follow the $T - X$ family of distributions defined in (2.4), then the *Shannon entropy* of the random variable X_g, η_{X_g}, is given by

$$\eta_{X_g} = \log\left(\frac{\pi}{b}\right) + 2E\left\{\log\left(F\left(X_g\right)\right)\right\} + 2\sum_{k=1}^{\infty} C_k E\left\{\left[F\left(X_g\right)\right]^{2k}\right\} + E\left\{\log\left(q_{X_f}\left(S(T)\right)\right)\right\} + \eta_T \tag{2.9}$$

where $C_k = (-1)^k(2\pi)^{2k}B_{2k}/(2k(2k)!)$ and B_k is the Bernoulli number, F is the CDF of a random variable X_f, $S(T) = \cot^{-1}\left((a-T)/b\right)/\pi$, $q_{X_f}(\lambda) = 1/f\left(F^{-1}(\lambda)\right) = 1/f(Q_{X_f}(\lambda))$ is the quantile density function of X_f, and η_T is the *Shannon entropy* of the random variable T with PDF $r(t)$.

Proof: The random variable$T = a - b\cot\left(\pi F(X_g)\right)$has the PDF$r(t)$, and the random variable $X_g = F^{-1}\{S(T)\} = Q_{X_f}(S(T))$ has the PDF $g(x)$. The function $g(x)$ in (2.3) becomes

$$g\left(X_g\right) = \pi b\csc^2\left(\pi F\left(X_g\right)\right) f(F^{-1}\{S(T)\})r(T)$$
$$= \pi b\left(1/\sin^2\left(\pi F(X_g)\right)\right)(1/q_{X_f}(S(T)))\, r(T)$$

, then $-\log(g(X_g)) = -\log(\pi b) + 2\log(\sin(\pi F(X_g))) + \log(q_{X_f}(S(T))) - \log(r(T))$ and by taking the expectation for the both sides, we get

$$E\left(-\log\left(g\left(X_g\right)\right)\right) = -\log(\pi b) + 2E\left[\log\left(\sin\left(\pi F\left(X_g\right)\right)\right)\right] + E\left\{\log\left(q_{X_f}\left(S(T)\right)\right)\right\} + \eta_T \tag{2.10}$$

By using the logarithm of sine series expansion from Jeffrey and Zwillinger (2007, p. 55)

$$\log(\sin(\pi u)) = \log(\pi u) + \sum_{k=1}^{\infty} C_k u^{2k} \tag{2.11}$$

where $C_k = (-1)^k(2\pi)^{2k}B_{2k}/(2k(2k)!)$, B_k is the Bernoulli number, and $0 < u < \pi$.

This implies

$$E\left[\log\left(\sin\left(\pi F\left(X_g\right)\right)\right)\right] = \log(\pi) + E\left\{\log\left(F\left(X_g\right)\right)\right\} + \sum_{k=1}^{\infty} C_k E\left\{\left[F\left(X_g\right)\right]^{2k}\right\} \tag{2.12}$$

Now substitute (2.12) in (2.10) to get the result in (2.9).

Theorem 3: Let $a = 0$ and the PDF $f(x)$ of the random variable X_f be symmetric about $x = d$. The PDF $g(x)$ of the random variable X_g is symmetric about $x = d$ if and only if the PDF $r(t)$ of the random variable T is symmetric about$t = 0$.

Proof: Suppose that the PDF $f(x)$ is symmetric about $x = d$, then for any real number x we have $f(x + d) = f(d - x)$ and $F(x + d) = 1 - F(d - x)$.
Suppose that the PDF $g(x)$ is symmetric about $x = d$, then for any real number x we have $g(x + d) = g(d - x)$ and $G(x + d) = 1 - G(d - x)$. Thus, from (1.8) we have $G(x + d) = R\{-b\cot(\pi F(x + d))\}$ and $G(d - x) = R\{-b\cot(\pi F(d - x))\}$. This implies $R\{-b\cot(\pi F(x + d))\} = 1 - R\{-b\cot(\pi F(d - x))\}$, now since $f(x)$ is symmetric about $x = d$, that is, $F(x + d) = 1 - F(d - x)$, and since $\cot(\pi - \theta) = -\cot(\theta)$, we get
$R\{-b\cot(\pi F(x + d))\} = R\{-b\cot(\pi[1 - F(d - x)])\} = R\{b\cot(\pi F(d - x))\}$
and then $R\{b\cot(\pi F(d - x))\} = 1 - R\{-b\cot(\pi F(d - x))\}$. Now since $T_{X_g} = -b\cot(\pi F(X_g))$, let $t_{d-x} = -b\cot(\pi F(d - x))$, and we obtain $R\{0 - t_{d-x}\} = 1 - R\{0 + t_{d-x}\}$. Hence, the PDF $r(t)$ of the random variable T is symmetric about $t = 0$.

Conversely, suppose that the PDF $r(t)$ is symmetric about $t = 0$, then for any real number t, $r(t) = r(-t)$ and $R(t) = 1 - R(-t)$. Let $t_{x+d} = -b\cot(\pi F(x + d))$, so $R\{-b\cot(\pi F(x + d))\} = R\{t_{x+d}\} = 1 - R\{-t_{x+d}\} = 1 - R\{b\cot(\pi F(x + d))\}$. From (2.1), we have $G(x) = R\{-b\cot(\pi F(x))\}$, so $G(x + d) = R\{-b\cot(\pi F(x + d))\} = R\{t_{x+d}\}$, and $1 - G(d - x) = 1 - R\{-b\cot(\pi F(d - x))\}$. Now since $f(x)$ is symmetric about $x = d$ and since $\cot(\pi - \theta) = -\cot(\theta)$, then $1 - G(d - x) = 1 - R\{b\cot(\pi F(x + d))\} = 1 - R\{-t_{x+d}\}$, this implies $G(x + d) = 1 - G(d - x)$. Hence, the PDF $g(x)$ of the random variable X_g is symmetric about $x = d$.
Conversely, suppose that the PDF $r(t)$ is symmetric about $t = 0$, then $r(t) = r(-t)$ and $R(t) = 1 - R(-t)$, let $t_{x+d} = -b\cot(\pi F(x + d))$, so $R\{-b\cot(\pi F(x + d))\} = R\{t_{x+d}\} = 1 - R\{-t_{x+d}\} = 1 - R\{b\cot(\pi F(x + d))\}$.From (10), we have $G(x) = R\{-b\cot(\pi F(x))\}$, so $G(x + d) = R\{-b\cot(\pi F(x + d))\} = R\{t_{x+d}\}$, and $1 - G(d - x) = 1 - R\{-b\cot(\pi F(d - x))\}$. Now since $f(x)$ is symmetric about $x = d$ and since $\cot(\pi - \theta) = -\cot(\theta)$, then $1 - G(d - x) = 1 - R\{b\cot(\pi F(x + d))\} = 1 - R\{-t_{x+d}\}$, this implies $G(x + d) = 1 - G(d - x)$. Hence, the PDF $g(x)$ of the random variable X_g is symmetric about $x = d$.

Theorem 4: Let the random variable X_f with PDF $f(x)$ and CDF $F(x)$ has the non-central n^{th} moment $E[X_f^n] \leq E[|X_f|^n] < \infty$, and the random variable T with PDF $r(t)$ and CDF $R(t)$ has the non-central n^{th} moment $E[T_f^n] \leq E[|T|_f^n] < \infty$, where $-\infty < T < \infty$, then the random variable X_g with PDF $g(x)$ and CDF $G(x)$ defined in (2.4) has the non-central n^{th} moment $E[X_g^n] \leq E[|X_g|^n] < \infty$ and satisfies the following

$$E[|X_g|^n] < \pi E[|X_f|^n]\left(\frac{3}{2b}\int_b^\infty tr(t)dt - \frac{3a}{2b} + \left(\frac{3a}{2b} + \frac{4}{\pi}\right)R(b) - \frac{8}{\pi}R(-b)\right) \qquad (2.13)$$

Proof: By definition,

$$E\left[|X|_f^n\right]=\int_{-\infty}^{\infty}|x|^n f(x)dx=\int_{-\infty}^{F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}}|x|^n f(x)dx+\int_{F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}}^{\infty}|x|^n f(x)dx$$
$$\geq\int_{F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}}^{\infty}|x|^n f(x)dx$$
$$\geq\left|F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}\right|^n\int_{F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}}^{\infty}f(x)dx$$
$$=\left|F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}\right|^n\left(\frac{1}{\pi}\left[\pi-\cot^{-1}\left(-\frac{T-a}{b}\right)\right]\right)$$

This implies, $\left|F^{-1}\left\{\cot^{-1}\left((a-T)/b\right)/\pi\right\}\right|^n\leq E\left[|X|_f^n\right]([\pi-\cot^{-1}(-(T-a)/b)]/\pi)^{-1}$, since $\cot^{-1}(u)=\pi-\cot^{-1}(-u)$, we get $\left|F^{-1}\left\{\cot^{-1}\left((a-T)/b\right)/\pi\right\}\right|^n\leq\pi E\left[|X|_f^n\right]\left(\cot^{-1}\left((T-a)/b\right)\right)^{-1}$. By Polyanin and Manzhirov (2008, p. 680) and Jeffrey and Zwillinger (2007, p. 61), $\cot^{-1}\left((T-a)/b\right)$ can be written as

$$\cot^{-1}\left(\frac{T-a}{b}\right)=\begin{cases}\sum_{k=0}^{\infty}\frac{(-1)^k b^{2k+1}}{(2k+1)}(T-a)^{-(2k+1)} & ;|T-a|>b\\ \frac{\pi}{2}-\sum_{k=0}^{\infty}\frac{(-1)^k b^{-(2k+1)}}{(2k+1)}(T-a)^{2k+1} & ;|T-a|\leq b\end{cases}\tag{2.14}$$

When $|T-a|\leq b$, we have $-b^{2k+1}\leq(T-a)^{2k+1}\leq b^{2k+1}$ for all $k=0,1,2,...$, and $-b^{2k+1}\leq-(T-a)^{2k+1}\leq b^{2k+1}$, then $\cot^{-1}\left((T-a)/b\right)=\pi/2+\sum_{k=0}^{\infty}(-1)^k b^{-(2k+1)}\left(-(T-a)\right)^{2k+1}/(2k+1)\geq\pi/2-\sum_{k=0}^{\infty}(-1)^k/(2k+1)=\pi/2-\cot^{-1}(1)=\pi/2-\pi/4=\pi/4$.

Hence, $\left(\cot^{-1}\left((T-a)/b\right)\right)^{-1}\leq 4/\pi$.

When $|T-a|>b$, we have $(T-a)^{2k+1}>b^{2k+1}$ or $(T-a)^{2k+1}<-b^{2k+1}$ for all $k=0,1,2,...$, so $b^{2k+1}/(T-a)^{2k+1}<1$ or $b^{2k+1}/(T-a)^{2k+1}>-1$ for all $k=0,1,2,....$

If $b^{2k+1}/(T-a)^{2k+1}<1$, we have

$$\sum_{k=0}^{\infty}\frac{(-1)^k b^{2k+1}}{(2k+1)}(T-a)^{-(2k+1)}=\frac{b}{(T-a)}-\frac{b^3}{3(T-a)^3}+\frac{b^5}{5(T-a)^5}-\frac{b^7}{7(T-a)^7}+\cdots$$
$$>\frac{b}{(T-a)}-\frac{b^3}{3(T-a)^3}=b\frac{3(T-a)^2-b^2}{3(T-a)^3}=b\frac{2(T-a)^2+(T-a)^2-b^2}{3(T-a)^3}$$
$$>b\frac{2(T-a)^2}{3(T-a)^3}=\frac{2b}{3(T-a)}.$$

Hence, $\left(\cot^{-1}\left((T-a)/b\right)\right)^{-1} < 3(T-a)/2b$.

If $(T-a)^{2k+1} < -b^{2k+1}$, then $(T-a)^{-(2k+1)} > -b^{-(2k+1)}$, and we have

$$\cot^{-1}\left(\frac{(T-a)}{b}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k b^{2k+1}(T-a)^{-(2k+1)}}{(2k+1)}$$
$$> -\sum_{k=0}^{\infty}(-1)^k/(2k+1) = -\cot^{-1}(1) = -\pi/4.$$

Hence, $\left(\cot^{-1}\left((T-a)/b\right)\right)^{-1} < -4/\pi$.
Now,

$$E\left\{\left(\left[\cot^{-1}\left(\frac{T-a}{b}\right)\right]\right)^{-1}\right\} = \int_{-\infty}^{\infty}\left(\left[\cot^{-1}\left(\frac{T-a}{b}\right)\right]\right)^{-1} r(t)dt = I_1 + I_2 + I_3$$

where

$$I_1 = \int_{-\infty}^{-b}\left(\left[\cot^{-1}\left(\frac{t-a}{b}\right)\right]\right)^{-1} r(t)dt < -\frac{4}{\pi}\int_{-\infty}^{-b} r(t)dt = -\frac{4}{\pi}R(-b)$$

$$I_2 = \int_{-b}^{b}\left(\left[\cot^{-1}\left(\frac{t-a}{b}\right)\right]\right)^{-1} r(t)dt < \frac{4}{\pi}\int_{-b}^{b} r(t)dt = \frac{4}{\pi}\left(R(b) - R(-b)\right)$$

and

$$I_3 = \int_{b}^{\infty}\left(\left[\cot^{-1}\left(\frac{t-a}{b}\right)\right]\right)^{-1} r(t)dt$$
$$< \frac{3}{2b}\int_{b}^{\infty}(t-a)r(t)dt$$
$$= \frac{3}{2b}\int_{b}^{\infty} tr(t)dt - \frac{3a}{2b}\int_{b}^{\infty} r(t)dt = \frac{3}{2b}\int_{b}^{\infty} tr(t)dt - \frac{3a}{2b}\left(1 - R(b)\right).$$

So,

$$\left|F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}\right|^n \le \pi E\left[|X|_f^n\right]\left(\cot^{-1}\left(\frac{T-a}{b}\right)\right)^{-1}$$

becomes

$$E\left\{\left|F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\frac{a-T}{b}\right)\right\}\right|^n\right\} \le \pi E\left[|X|_f^n\right] E\left\{\left(\cot^{-1}\left(\frac{T-a}{b}\right)\right)^{-1}\right\}$$

and from remark (b) above, we get the result in (2.14).

Suppose $S(x) = a - b\cot\left(\pi F(x)\right)$ and $D(x) = \exp(-(S(x)-\mu)/\sigma)$. **Table 1** shows a list of some examples of subfamilies of the $T-X$ family of distributions defined in (2.4) using the tangent function for different T random variables.

Table 1: Some examples of the $T-X$ family of distributions using the tangent function derived from different T distributions

The name of PDF $r(t)$	The PDF $r(t)$	The PDF $g(x)$ of the $T-X$ family of distributions defined in (2.1)
Normal (Gaussian)	$\frac{1}{\sigma}\phi\left(\frac{t-\mu}{\sigma}\right)$	$\frac{\pi b}{\sigma}\csc^2\left(\pi F(x)\right)f(x)\phi\left(\frac{S(x)-\mu}{\sigma}\right)$
Cauchy	$\frac{1}{\pi\lambda\left(1+\left(\frac{t-\theta}{\lambda}\right)^2\right)}$	$\frac{b}{\lambda}\csc^2\left(\pi F(x)\right)f(x)\frac{1}{\left(1+\left(\frac{S(x)-\theta}{\lambda}\right)^2\right)}$
Extreme Value Type I (Gumbel)	$\frac{1}{\sigma}\exp\left(-e^{-\left(\frac{t-\mu}{\sigma}\right)}\right)e^{-\left(\frac{t-\mu}{\sigma}\right)}$	$\frac{\pi b}{\sigma}\csc^2\left(\pi F(x)\right)f(x)\exp\left(-D(x)\right)D(x)$
Laplace	$\frac{1}{2\sigma}\exp\left(-\left\|\frac{t-\mu}{\sigma}\right\|\right)$	$\frac{\pi b}{2\sigma}\csc^2\left(\pi F(x)\right)f(x)\exp\left(-\left\|\frac{S(x)-\mu}{\sigma}\right\|\right)$
Logistic	$\frac{1}{\sigma}\frac{\exp\left(-\left(\frac{t-\mu}{\sigma}\right)\right)}{\left\{1+\exp\left(-\left(\frac{t-\mu}{\sigma}\right)\right)\right\}^2}$	$\frac{\pi b}{\sigma}\csc^2\left(\pi F(x)\right)f(x)\frac{D(x)}{\left(1+D(x)\right)^2}$
Student's t	$\frac{1}{\sqrt{v}B\left(\frac{1}{2},\frac{v}{2}\right)}\left(1+\frac{t^2}{v}\right)^{-(v+1)/2}$	$\frac{\pi b}{\sqrt{v}B\left(\frac{1}{2},\frac{v}{2}\right)}\csc^2\left(\pi F(x)\right)f(x)\left(1+\frac{\left(S(x)\right)^2}{v}\right)^{-(v+1)/2}$
Generalized Error (Generalized Normal)	$\frac{1}{2^{k+1}\sigma\Gamma(k+1)}e^{-\frac{1}{2}\left\|\frac{t-\mu}{\sigma}\right\|^{\frac{1}{k}}}$	$\frac{\pi b}{2^{k+1}\sigma\Gamma(k+1)}\sec^2\left(\pi F(x)\right)f(x)e^{-\frac{1}{2}\left\|\frac{S(x)-\mu}{\sigma}\right\|^{\frac{1}{k}}}$

3. Some sub-families of the $T-X$ family of distributions with different T distributions

The subfamilies from the $T-X$ family of distributions defined in (2.4) can be gotten in two different ways: fix the random variable X_f distribution and change the random variable T distributions, and the other fix the variable T distribution and change the random variable X_f distributions.

In **Table 1** above, the random variable X_f has been fixed, and by changing the random variable T distributions, one can be gotten several such sub-families. For example: let the random variable T be normally distributed, we generate a sub-family of Normal$-X$ distributions.

In the following sub-sections, some properties of the following sub-families will be discussed: Normal$-X$ sub-family, Cauchy$-X$ sub-family, and Generalized Error$-X$ sub-family.

3.1 The Normal$-X$ Sub-Family

Let the random variable T follow a normal distribution with location parameter μ and scale parameter σ, $T\sim N(\mu,\sigma)$, then the PDF $r(t;\mu,\sigma)=\left(1/\sqrt{2\pi}\sigma\right)\exp\left(-\left((t-\mu)/\sigma\right)^2/2\right)=(1/\sigma)\phi\left((t-\mu)/\sigma\right)$, $t\in\mathbb{R},\mu\in\mathbb{R},\sigma>0$, and the CDF $R(t)=\Phi\left((t-\mu)/\sigma\right)$, where $\Phi(y)=\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{y}\exp(-t^2/2)\,dt$. By substituting in (2.3), the PDF $g(x)$ of the Normal$-X$ sub-family is defined as

$$g(x)=\frac{\pi}{v}\csc^2\left(\pi F(x)\right)f(x)\phi\left(\frac{1}{v}\left[\gamma-\cot\left(\pi F(x)\right)\right]\right) \tag{3.1}$$

where $\gamma = c/b$, $c = a - \mu \in \mathbb{R}$ and $\nu = \sigma/b > 0$, by substituting in (2.4), the CDF $G(x)$ of the Normal$-X$ sub-family is defined as

$$G(x) = \Phi\left(\frac{1}{\nu}\left[\gamma - \cot\left(\pi F(x)\right)\right]\right) \tag{3.2}$$

Lemma 1: Let $0 < \lambda < 1$, the *quantile function* of the Normal$-X$ sub-family of distributions defined in (3.2) is given by

$$Q_{X_g}(\lambda) = F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\left(\gamma - \nu\Phi^{-1}(\lambda)\right)\right)\right\} \tag{3.3}$$

where $F^{-1}(\lambda) = Q_{X_f}(\lambda)$ is the *quantile function* of the random variable X_f with CDF $F(x)$.

Proof: By equation (2.7) in **Theorem 1** above, since the quantile of the random variable T with CDF $R(t)$ is the quantile of the normal distribution with parameters μ and σ, and it is given by $Q_T(\lambda) = R^{-1}(\lambda) = \mu + \sigma\Phi^{-1}(\lambda)$, hence it can be easily gotten the result in (3.3).

Lemma 2: The *Shannon entropy* of the Normal$-X$ sub-family of distributions defined in (3.2), η_{X_g}, is given by

$$\begin{aligned}\eta_{X_g} = &\left(\frac{3}{2}\log(\pi) + \frac{1}{2}\log(2) + \frac{1}{2}\right) + \log(\nu) + 2E\left\{\log\left(F\left(X_g\right)\right)\right\} \\ &+2\sum_{k=1}^{\infty} C_k E\left\{\left[F\left(X_g\right)\right]^{2k}\right\} + E\left\{\log\left(q_{X_f}\left(S(T)\right)\right)\right\}\end{aligned} \tag{3.4}$$

where $C_k = (-1)^k(2\pi)^{2k}B_{2k}/(2k(2k)!)$ and B_k is the Bernoulli number, $F(x)$ is the CDF of a random variable X_f, and $S(T) = \cot^{-1}\left((a - T)/b\right)/\pi$, $q_{X_f}(\lambda) = 1/f\left(F^{-1}(\lambda)\right) = 1/f(Q_{X_f}(\lambda))$ is the quantile density function of X_f.

Proof: Since the random variable T has the normal distribution with parameters μand σ, then its *Shannon entropy*, η_T, is defined as $\log(2\pi\sigma^2 e)/2$. Now substitute in (2.9) we get the result (3.4).

Lemma 3: Let the random variable X_f with PDF $f(x)$ and CDF $F(x)$ has the non-central n^{th} moment $E\left[X_f^n\right] \le E\left[\left|X_f\right|^n\right] < \infty$, and the random variable Tnormal distribution with parameters μ and σ, $T\sim N(\mu,\sigma)$, then the random variable X_g with PDF $g(x)$ and CDF $G(x)$ defined in (3.2) has the non-central n^{th} moment $E\left[X_g^n\right] \le E\left[\left|X_g\right|^n\right] < \infty$ and satisfies the following

$$E\left[\left|X_g\right|^n\right] < \pi E\left[\left|X_f\right|^n\right]\left(\frac{3\sigma}{2b}\phi\left(\frac{b-\mu}{\sigma}\right) + \frac{3}{2b}(\mu - a)\Phi(-b) - \frac{4}{\pi}\Phi(b)\right) \tag{3.5}$$

Proof: Since $T\sim N(\mu,\sigma)$, then

$$\int_b^{\infty} tr(t)dt = \frac{1}{\sqrt{2\pi}\sigma}\int_b^{\infty} te^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$
$$= \frac{1}{\sqrt{2\pi}}\int_b^{\infty} \left(\frac{t-\mu}{\sigma}\right) e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt + \frac{1}{\sqrt{2\pi}\sigma}\mu\int_b^{\infty} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$
$$= \frac{\sigma}{\sqrt{2\pi}}\int_{\left(\frac{b-\mu}{\sigma}\right)}^{\infty} ye^{-\frac{1}{2}y^2}dy + \mu\int_b^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$
$$= \sigma\,\phi\left(\frac{b-\mu}{\sigma}\right) + \mu\big(1-\Phi(b)\big) < \infty$$

Hence, by substituting in (2.13), we obtain the result in (3.5).

One example on this family, let the random variable X_f follow Generalized Hyperbolic Secant (Generalized inverse-hyperbolic cosine) distribution with location parameter $-\infty < \alpha < \infty$ and scale parameter $\beta > 0$, its PDF is given by $f(x;\alpha,\beta) = (1/2\beta)\operatorname{sech}\big[(\pi/2)\big((x-\alpha)/\beta\big)\big]$ where $-\infty < x < \infty$, and its CDF is given by $F(x) = (1/2) + \tan^{-1}\big(\sinh\big[(\pi/2)\big((x-\alpha)/\beta\big)\big]\big)/\pi$, and its *quantile function* is given by $F^{-1}(\lambda;\alpha,\beta) = Q_{X_f}(\lambda;\alpha,\beta) = (\pi/2)\beta\sinh^{-1}\{\tan[\pi(\lambda - 1/2)]\} + \alpha$.

Now since $\cot^{-1}(-y) = \pi/2 + \tan^{-1}(y)$, $\cot^{-1}(-y) = \pi - \cot^{-1}(y)$, and $\csc(\cot^{-1}(y)) = \sqrt{1+y^2}$, from (3.1) we get

$$NGHS(x;\boldsymbol{\theta}) = \frac{\pi}{2\beta\nu}\cosh\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right]\phi\left(\frac{1}{\nu}\left(\gamma + \sinh\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right]\right)\right) \quad (3.6)$$

where $x \in \mathbb{R}$, $\alpha,\gamma \in \mathbb{R}$ and $\beta,\nu > 0$, and $\boldsymbol{\theta} = (\gamma,\nu,\alpha,\beta)'$. The CDF of $NGHS$ is given by $G(x;\boldsymbol{\theta}) = \Phi\big(\big[\gamma + \sinh\big[(\pi/2)\big((x-\alpha)/\beta\big)\big]\big]/\nu\big)$. The random variable X_g with PDF in (3.6) is said to be follow a four-parameter Normal-Generalized Hyperbolic Secant distribution ($NGHS$).

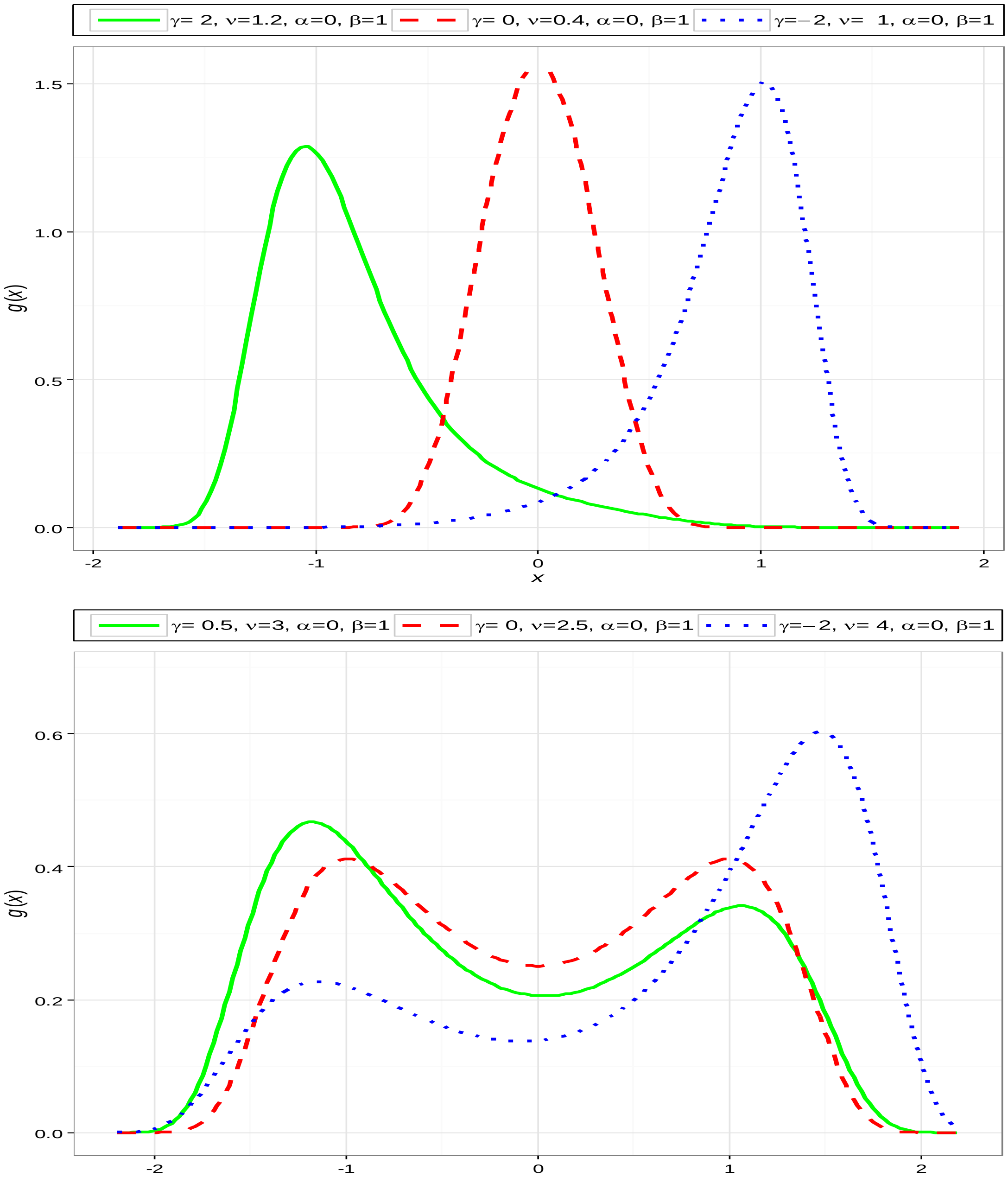
Plots in **Figure 1** show the $NGHS$ density function for different parameter values, the distribution can be symmetric, right skewed, left skewed, unimodal or bimodal.

Lemma 4: Let $0 < \lambda < 1$, the *quantile function* of the $NGHS$distribution defined in (3.6) is given by

$$Q_{NGHS}(\lambda) = \frac{2\beta}{\pi}\sinh^{-1}\{\nu\Phi^{-1}(\lambda) - \gamma\} + \alpha \quad (3.7)$$

Proof: By equation (3.3) in **Lemma 1** above, since the quantile of the random variable X_f with CDF $F(x)$ is the quantile of the Generalized Hyperbolic Secant distribution with parameters α and β, and it is given by $F^{-1}(\lambda;\alpha,\beta) = Q_{X_f}(\lambda;\alpha,\beta) = (\pi/2)\beta\sinh^{-1}\{\tan[\pi(\lambda - 1/2)]\} + \alpha$, since $\tan^{-1}(y) = \cot^{-1}(-y) - \pi/2$, hence it can be easily gotten the result in (3.7).

Figure 1 The PDF of Normal-Generalized hyperbolic secant distribution for various values of γ and ν.



Lemma 5: Let the random variable X_g has $NGHS$ distribution, then the non-central n^{th} moment $E\left[X_g^n\right]$ exists and satisfiesthe following inequality

$$E\left[\left|X_g\right|^n\right] < \pi\left(\sum_{i=0}^{n}\binom{n}{i}\beta^i\alpha^{n-i}|E_i|\right)m(a,b,\mu,\sigma) \tag{3.8}$$

where $E_i's$ are the Euler numbers, and

$m(a,b,\mu,\sigma) = \Big((3\sigma/2b)\phi\big((b-\mu)/\sigma\big) - (3c/2b)\Phi(-b) - (4/\pi)\Phi(b)\Big).$

Proof: $E\left[\left|X_f\right|^n\right] = \frac{1}{2\beta}\int_{-\infty}^{\infty} |x|^n \operatorname{sech}\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right] dx = \frac{1}{\beta}\int_0^{\infty} x^n \operatorname{sech}\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right] dx.$

Let $u = \frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right) \Longrightarrow x = \frac{2\beta}{\pi}u + \alpha \Longrightarrow dx = \frac{2\beta}{\pi}du$, then

$$E\left[\left|X_f\right|^n\right] = \frac{2}{\pi}\int_0^{\infty} \left(\frac{2\beta}{\pi}u + \alpha\right)^n \operatorname{sech}(u)\, du.$$

By binomial theorem:

$$\left(\frac{2\beta}{\pi}u + \alpha\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{2\beta}{\pi}u\right)^i \alpha^{n-i}$$

So,

$$E\left[\left|X_f\right|^n\right] = \frac{2}{\pi}\int_0^{\infty} \sum_{i=0}^{n} \binom{n}{i} \left(\frac{2\beta}{\pi}u\right)^i \alpha^{n-i} \operatorname{sech}(u)\, du$$

$$= \frac{1}{\pi}\sum_{i=0}^{n} \binom{n}{i} \left(\frac{2\beta}{\pi}\right)^i \alpha^{n-i} \int_0^{\infty} u^i \operatorname{sech}(u)\, du$$

$$= \frac{2}{\pi}\sum_{i=0}^{n} \binom{n}{i} \left(\frac{2\beta}{\pi}\right)^i \alpha^{n-i} \int_0^{\infty} u^i \operatorname{sech}(u)\, du$$

But

$$\int_0^{\infty} u^i \operatorname{sech}(u)\, du = \frac{1}{2^i}\Gamma(i+1)\sum_{m=0}^{\infty}(-1)^m\left(\frac{2}{2m+1}\right)^{i+1} = \left(\frac{\pi}{2}\right)^{i+1}|E_i|$$

where $E_i's$ are the Euler numbers. For the odd-indexed all the Euler numbers are all zero, and for the even-indexed $E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, \ldots$.

So,

$$E\left[\left|X_f\right|^n\right] = \frac{2}{\pi}\sum_{i=0}^{n} \binom{n}{i} \left(\frac{2\beta}{\pi}\right)^i \alpha^{n-i} \left(\frac{\pi}{2}\right)^{i+1} |E_i| = \sum_{i=0}^{n} \binom{n}{i} \beta^i \alpha^{n-i} |E_i| < \infty.$$

Hence, by substituting in (3.5), we obtain the result in (3.8).

3.2 The Gumbel$-X$ Sub-Family

Let the random variable T follow a Gumbel distribution (Extreme value type I distribution) with location parameter μ and scale parameter σ, $T \sim G(\mu, \sigma)$, then the PDF $r(t; \mu, \sigma) = (1/\sigma)\exp(-\exp(-(t-\mu)/\sigma))\exp(-(t-\mu)/\sigma)$, $t \in (-\infty, \infty)$, $\mu \in \mathbb{R}$, $\sigma > 0$, and the CDF $R(t) = \exp(-\exp(-(t-\mu)/\sigma))$. By substituting in (2.3), the PDF $g(x)$ of the Gumbel$-X$ sub-family is defined as

$$g(x) = \frac{\pi}{\nu}\csc^2\big(\pi F(x)\big) f(x) e^{-e^{-\frac{1}{\nu}[\gamma - \cot(\pi F(x))]}} e^{-\frac{1}{\nu}[\gamma - \cot(\pi F(x))]} \tag{3.9}$$

where $\gamma = c/b$, $c = a - \mu \in \mathbb{R}$ and $\nu = \sigma/b > 0$, by substituting in (2.4), the CDF $G(x)$ of the Gumbel$-X$ sub-family is defined as

$$G(x) = e^{-e^{-\frac{1}{\nu}[\gamma - \cot(\pi F(x))]}} \tag{3.10}$$

Lemma 6: Let $0 < \lambda < 1$, the *quantile function* of the Gumbel$-X$ sub-family of distributions defined in (3.10) is given by

$$\begin{aligned} Q_{X_g}(\lambda) &= Q_{X_f}\left(\frac{1}{\pi}\cot^{-1}(\gamma + \nu\log(-\log(\lambda)))\right) \\ &= F^{-1}\left\{\frac{1}{\pi}\cot^{-1}(\gamma + \nu\log(-\log(\lambda)))\right\} \end{aligned} \tag{3.11}$$

where $F^{-1}(\lambda) = Q_{X_f}(\lambda)$ is the *quantile function* of the random variable X_f with CDF $F(x)$.

Proof: By equation (2.7) in **Theorem 1** above, since the quantile of the random variable T with CDF $R(t)$ is the quantile of the Gumbel distribution with parameters μ and σ, and it is given by $Q_T(\lambda) = R^{-1}(\lambda) = \mu - \sigma\log(-\log(\lambda))$, hence it can be easily gotten the result in (3.11).

Lemma 7: The *Shannon entropy* of the Gumbel$-X$ sub-family of distributions defined in (3.10), η_{X_g}, is given by

$$\begin{aligned} \eta_{X_g} &= (\log(\pi) + \gamma^* + 1) + \log(\gamma) + 2E\left\{\log\left(F(X_g)\right)\right\} \\ &+2\sum_{k=1}^{\infty} C_k E\left\{\left[F(X_g)\right]^{2k}\right\} + E\left\{\log\left(q_{X_f}(S(T))\right)\right\} \end{aligned} \tag{3.12}$$

where $C_k = (-1)^k(2\pi)^{2k}B_{2k}/(2k(2k)!)$ and B_k is the Bernoulli number, $\gamma^* \approx 0.5772$ is the Euler–Mascheroni constant, $F(x)$ is the CDF of a random variable X_f, $S(T) = \cot^{-1}\left((a-T)/b\right)/\pi$, and $q_{X_f}(\lambda) = 1/f\left(F^{-1}(\lambda)\right) = 1/f(Q_{X_f}(\lambda))$ is the quantile density function of X_f.

Proof: Since the random variable T has the Gumbel distribution with parameters θ and λ, then its *Shannon entropy*, η_T, is defined as $\log(\sigma) + \gamma^* + 1$, where $\gamma^* \approx 0.5772$ is the Euler–Mascheroni constant. Now substitute in (2.9) we get the result (3.12).

Lemma 8: Let the random variable X_f with PDF $f(x)$ and CDF $F(x)$ has the non-central n^{th} moment $E\left[X_f^n\right] \leq E\left[\left|X_f\right|^n\right] < \infty$, and the random variable T Gumbel distribution with parameters μ and σ, $T \sim G(\mu, \sigma)$, then the random variable $X_g$with PDF $g(x)$ and CDF $G(x)$ defined in (3.10) has the non-central n^{th} moment $E\left[X_g^n\right] \leq E\left[\left|X_g\right|^n\right] < \infty$ and satisfies the following

$$E\left[\left|X_g\right|^n\right] \leq \pi E\left[\left|X_f\right|^n\right] m(a, b, \mu, \sigma) \tag{3.13}$$

where

$$m(a,b,\mu,\sigma) = \frac{3}{2}\left\{\frac{\mu}{b} - \frac{\sigma}{b} - \sigma e^{-e^{-\frac{b-\mu}{\sigma}}} - \frac{a}{b} - \frac{\sigma^2}{b}Ei\left(-e^{-\frac{b-\mu}{\sigma}}\right)\right\} + \left(\frac{3a}{2b} + \frac{4}{\pi}\right)R(b) - \frac{8}{\pi}R(-b)$$

, $\gamma^* \approx 0.5772$ is the Euler–Mascheroni constant and $Ei(y) = \int_{-\infty}^{y} e^s s^{-1} ds\,; y < 0$ is the exponential integral function.

Proof: Since $T \sim G(\mu,\sigma)$, then

$$\int_b^\infty tr(t)dt = E(T) - \int_{-\infty}^b tr(t)dt = \mu - \gamma\sigma - \int_{-\infty}^b tr(t)dt.$$

But,

$$\int_{-\infty}^b tr(t)dt = \sigma \int_{-\infty}^b \left(\frac{t-\mu}{\sigma}\right) e^{-e^{-\frac{t-\mu}{\sigma}}} e^{-\frac{t-\mu}{\sigma}} dt + \sigma \frac{\mu}{\sigma} \int_{-\infty}^b e^{-e^{-\frac{t-\mu}{\sigma}}} e^{-\frac{t-\mu}{\sigma}} dt$$

$$= \sigma^2 \int_{-\infty}^{\frac{b-\mu}{\sigma}} u e^{-e^{-u}} e^{-u} du + \sigma\mu \int_{-\infty}^{\frac{b-\mu}{\sigma}} e^{-e^{-u}} e^{-u} du$$

$$= \sigma(b-\mu) e^{-e^{-\frac{b-\mu}{\sigma}}} - \sigma^2 \int_{-\infty}^{\frac{b-\mu}{\sigma}} e^{-e^{-u}} du + \sigma\mu e^{-e^{-\frac{b-\mu}{\sigma}}}$$

$$= \sigma b e^{-e^{-\frac{b-\mu}{\sigma}}} - \sigma^2 \int_{-\infty}^{\frac{b-\mu}{\sigma}} e^{-e^{-u}} du = \sigma b e^{-e^{-\frac{b-\mu}{\sigma}}} + \sigma^2 \int_{-\infty}^{-e^{-\frac{b-\mu}{\sigma}}} y^{-1} e^{y} dy$$

$$= \sigma b e^{-e^{-\frac{b-\mu}{\sigma}}} + \sigma^2 Ei\left(-e^{-\frac{b-\mu}{\sigma}}\right)$$

Hence,

$$\int_b^\infty tr(t)dt = \mu - \sigma\left[\gamma + b e^{-e^{-\frac{b-\mu}{\sigma}}} + \sigma Ei\left(-e^{-\frac{b-\mu}{\sigma}}\right)\right] < \infty$$

Now, by substituting in (2.13), we obtain the result in (3.13). Knowing that, $Ei(y) = \int_{-\infty}^{y} e^s s^{-1} ds\,; y < 0$ is bounded functions Alzer (1997).

One example on this family, let the random variable X_f follow Generalized Hyperbolic Secant distribution with location parameter $-\infty < \alpha < \infty$ and scale parameter $\beta > 0$ as in **Section 3.1**. From (3.9) we get

$$GGHS(x;\boldsymbol{\theta}) = \frac{\pi}{2\beta\nu} \cosh\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right] e^{-e^{-\frac{1}{\nu}\left(\gamma+\sinh\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right]\right)}} e^{-\frac{1}{\nu}\left(\gamma+\sinh\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right]\right)} \quad (3.14)$$

where $x \in \mathbb{R}$, $\alpha, \gamma \in \mathbb{R}$ and $\beta, \nu > 0$, and $\boldsymbol{\theta} = (\gamma, \nu, \alpha, \beta)'$. The CDF of $GGHS$ is given by $G(x;\boldsymbol{\theta}) = G(x) = \exp\{-\exp(-[c + b\sinh[(\pi/2)((x-\alpha)/\beta)]]/\sigma)\}$. The random variable X_g with PDF in (3.14) is said to be follow a four-parameter Gumbel-Generalized Hyperbolic Secant distribution ($GGHS$).

Plots in **Figure 2** show the $GGHS$ density function for different parameter values, the distribution can be right skewed, left skewed, unimodal or bimodal.

Lemma 9: Let $0 < \lambda < 1$, the *quantile function* of the $GGHS$ distribution defined in (3.14) is given by

$$Q_{GGHS}(\lambda) = \alpha - \frac{2\beta}{\pi}\sinh^{-1}\{\gamma + \nu \log(-\log(\lambda))\} \tag{3.15}$$

Proof: By equation (3.11) in **Lemma 6** above, since the quantile of the random variable X_f with CDF $F(x)$ is the quantile of the Generalized Hyperbolic Secant distribution with parameters α and β, and it is given by $F^{-1}(\lambda;\alpha,\beta) = Q_{X_f}(\lambda;\alpha,\beta) = (\pi/2)\beta\sinh^{-1}\{\tan[\pi(\lambda - 1/2)]\} + \alpha$. Since $\tan^{-1}(y) = \cot^{-1}(-y) - \pi/2$and $\sinh^{-1}(-y) = -\sinh^{-1}(y)$, the result in (3.15) follows.

Lemma 10: Let the random variable X_g has $GGHS$ distribution, then the non-central n^{th} moment $E\left[X_g^n\right]$ exists and satisfiesthe following inequality

$$E\left[\left|X_g\right|^n\right] < \pi\left(\sum_{i=0}^{n}\binom{n}{i}\beta^i\alpha^{n-i}|E_i|\right)m(a,b,\mu,\sigma) \tag{3.16}$$

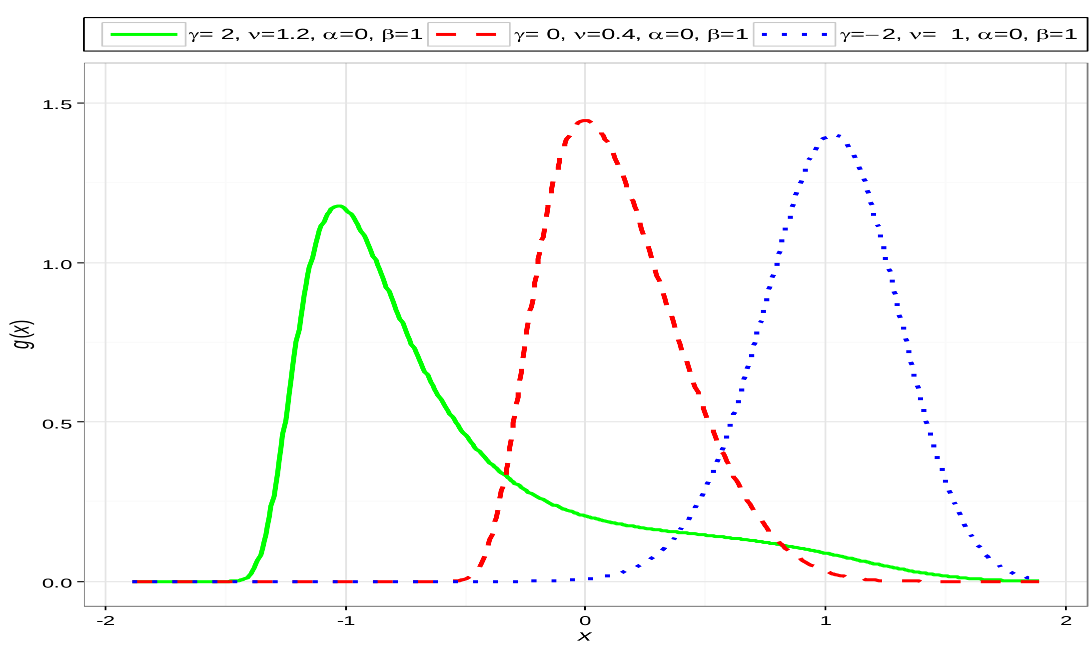
where $E_i's$ are the Euler numbers,

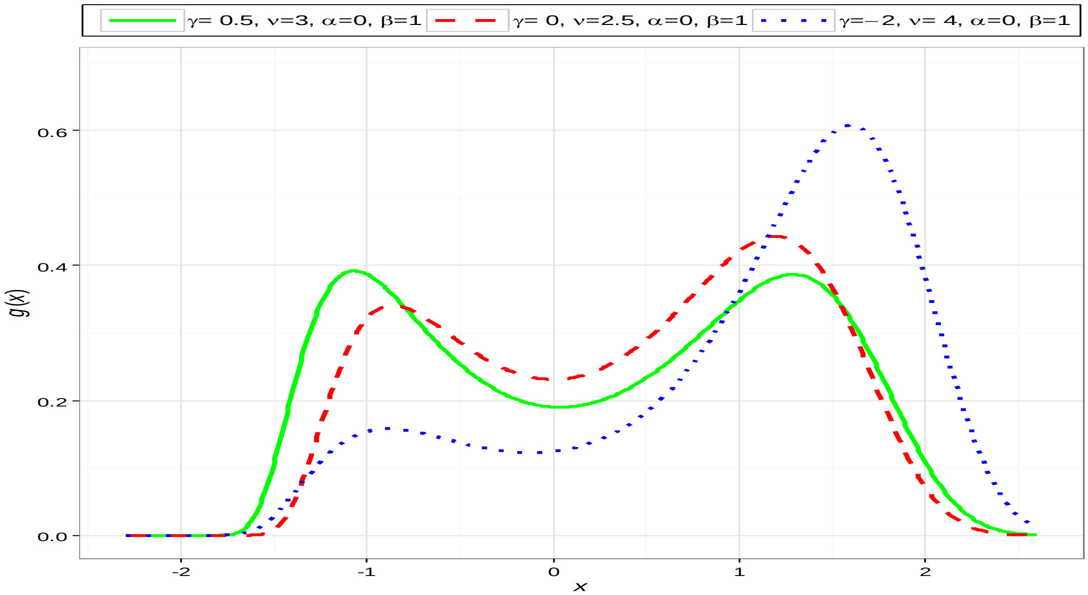
$$m(a,b,\mu,\sigma) = \frac{3}{2}\left\{\frac{\mu}{b} - \frac{\sigma\gamma^*}{b} - \sigma e^{-e^{-\frac{b-\mu}{\sigma}}} - \frac{a}{b} - \frac{\sigma^2}{b}Ei\left(-e^{-\frac{b-\mu}{\sigma}}\right)\right\} + \left(\frac{3a}{2b} + \frac{4}{\pi}\right)R(b) - \frac{8}{\pi}R(-b)$$

and $\gamma^*$is the Euler–Mascheroni constant, and$Ei(\cdot)$ is the exponential integral function.

Proof: By using the same steps used in proving **Lemma 5** above, and by substituting in (3.13), we obtain the result in (3.16).

Figure 2 The PDF of Gumbel-Generalized hyperbolic secant distribution for various values of γ and ν.





3.3 The Generalized Error$-X$ Sub-Family

Let the random variable T follow a Generalized Error Distribution (Generalized Normal Distribution or Exponential Power Distribution) with location parameter μ, scale parameter σ, and shape parameter k, $T \sim GE(\mu, \sigma, k)$, then the PDF $r(t;\mu,\sigma,k) = (1/(2^{k+1}\sigma\Gamma(k+1)))\exp\left\{-(1/2)|(t-\mu)/\sigma|^{\frac{1}{k}}\right\}$, $t \in (-\infty,\infty)$, $\mu \in \mathbb{R}$, $\sigma > 0, k > 0$, and the CDF $R(t) = (1/2)\left[1 + \left(\text{sgn}(t-\mu)/\Gamma(k)\right)\gamma\left(k, |(t-\mu)/\sigma|^{\frac{1}{k}}\right)\right]$, where $\gamma(s,y) = \int_0^y t^{s-1}e^{-t}dt$ is the incomplete gamma function.

Note: There are some special cases of the Generalized Error Distribution:
- If $k = 1/2$ then $GE(\mu,\sigma,1/2) \stackrel{d}{=} N(\mu,\sigma)$ which is the Normal Distribution.
- If $k = 1$ then $GE(\mu,\sigma,1) \stackrel{d}{=} L(\mu, 2\sigma)$ which is the Double Exponential (Laplace Distribution).
- If the limit $k \longrightarrow 0$, $GE(\mu,\sigma,k \longrightarrow 0) \stackrel{d}{=} U(\mu-\sigma,\mu+\sigma)$ which is the Uniform Distribution.

By substituting in (2.3), the PDF $g(x)$ of the Generalized Error$-X$ sub-family is defined as

$$g(x) = \frac{\pi}{2^{k+1}\nu\Gamma(k+1)}\csc^2\left(\pi F(x)\right) f(x) e^{-\frac{1}{2}\left|\frac{1}{\nu}\left[\gamma - \cot\left(\pi F(x)\right)\right]\right|^{\frac{1}{k}}} \tag{3.17}$$

where $\gamma = c/b$, $c = a - \mu \in \mathbb{R}$ and $\nu = \sigma/b > 0$, by substituting in (2.4), the CDF $G(x)$ of the Generalized Error $-X$ sub-family is defined as

$$G(x) = \frac{1}{2}\left[1 + \frac{\text{sgn}\left(\gamma - \cot\left(\pi F(x)\right)\right)}{\Gamma(k)}\gamma\left(k, \left|\frac{1}{\nu}\left[\gamma - \cot\left(\pi F(x)\right)\right]\right|^{\frac{1}{k}}\right)\right] \tag{3.18}$$

where $\gamma(s,y) = \int_0^y t^{s-1}e^{-t}dt$ is the lower incomplete gamma function.

Lemma 11: Let the random variable X_g follow thedistributions defined in (3.18), then:

a) If $k = 1/2$, we get the same Normal$-X$ sub-family of distributions defined in **Section 3.1**.
b) If $k = 1$, we get the Double Exponential (Laplace)$-X$ sub-family of distributions.
c) If the limit $k \longrightarrow 0$, we get Uniform Distribution$-X$ sub-family of distributions.

Lemma 12: Let $0 < \lambda < 1$, the *quantile function* of the Generalized Error$-X$ sub-family of distributions defined in (3.18) is given by

$$\begin{aligned} Q_{X_g}(\lambda) &= Q_{X_f}\left(\frac{1}{\pi}\cot^{-1}\left(\gamma - \nu\,\mathrm{sign}(2\lambda-1)\left(2Q_{gamma}(|2\lambda-1|,k)\right)^k\right)\right) \\ &= F^{-1}\left\{\frac{1}{\pi}\cot^{-1}\left(\gamma - \nu\,\mathrm{sign}(2\lambda-1)\left(2Q_{gamma}(|2\lambda-1|,k)\right)^k\right)\right\} \end{aligned} \tag{3.19}$$

where $Q_{gamma}(\lambda)$ is the *quantile function* of a gamma distribution, and $F^{-1}(\lambda) = Q_{X_f}(\lambda)$ is the *quantile function* of the random variable X_f with CDF $F(x)$.

Proof: By equation (2.7) in **Theorem 1** above, since the quantile of the random variable T with CDF $R(t)$ is the quantile of the Generalized Error distribution with parameters $\mu, \sigma,$ and $k,$ and it is given by $Q_T(\lambda) = R^{-1}(\lambda) = \sigma\,\mathrm{sign}(2\lambda-1)(2Q_{gamma}(|2\lambda - 1|, k))^{\,k} + \mu$, where $Q_{gamma}(\lambda)$ is the *quantile function* of gamma distribution, hence it can be easily gotten the result in (3.19).

Lemma 13: The *Shannon entropy* of the Generalized Error$-X$ sub-family of distributions defined in (3.18), η_{X_g}, is given by

$$\begin{aligned} \eta_{X_g} &= \log(\pi) + k + (k+1)\log(2) + \log\big(\Gamma(k+1)\big) + \log(\gamma) \\ &\quad +2E\left\{\log\left(F\left(X_g\right)\right)\right\} + 2\sum_{k=1}^{\infty} C_k E\left\{\left[F\left(X_g\right)\right]^{2k}\right\} + E\left\{\log\left(q_{X_f}\left(S(T)\right)\right)\right\} \end{aligned} \tag{3.20}$$

where $C_k = (-1)^k (2\pi)^{2k} B_{2k}/(2k(2k)!)$ and B_k is the Bernoulli number, $F(x)$ is the CDF of a random variable X_f, $S(T) = \cot^{-1}\big((a-T)/b\big)/\pi$, $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ is the gamma function, and $q_{X_f}(\lambda) = 1/f\big(F^{-1}(\lambda)\big) = 1/f(Q_{X_f}(\lambda))$ is the quantile density function of X_f.

Proof: Since the random variable T has the Generalized Error distribution with parameters $\mu, \sigma,$ and, then its *Shannon entropy*, η_T, is defined as $k + \log\Big(2^{k+1}\sigma\Gamma(k+1)\Big)$. Now substitute in (2.9) we get the result (3.20).

Lemma 14: Let the random variable $X_f$with PDF $f(x)$ and CDF $F(x)$ has the non-central n^{th} moment $E\left[X_f^n\right] \le E\left[\left|X_f\right|^n\right] < \infty$, and the random variable T Generalized Error distribution with parameters $\mu, \sigma,$ and k, $T \sim GE(\mu, \sigma, k)$, then the random variable $X_g$with PDF $g(x)$ and CDF $G(x)$ defined in (3.18) has the non-central n^{th} moment $E\left[X_g^n\right] \le E\left[\left|X_g\right|^n\right] < \infty$ and satisfiesthe following

$$E\left[\left|X_g\right|^n\right] < \begin{cases} \pi E\left[\left|X_f\right|^n\right] m_1(a,b,\mu,\sigma,k) & ;\text{if } b \geq \mu \\ \pi E\left[\left|X_f\right|^n\right] m_2(a,b,\mu,\sigma,k) & ;\text{if } b < \mu \end{cases} \quad (3.21)$$

where

$m_1(a,b,\mu,\sigma,k) =$

$= \frac{3}{4b\Gamma(k)}\left\{\sigma 2^k \Gamma\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \mu\Gamma\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right)\right\} - \frac{3a}{2b} + \left(\frac{3a}{2b} + \frac{4}{\pi}\right) R(b) - \frac{8}{\pi} R(-b),$

$m_2(a,b,\mu,\sigma,k) =$

$= \frac{3}{4b\Gamma(k)}\left\{\sigma 2^k \gamma_*\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \mu\gamma\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \frac{\mu}{2}\right\} - \frac{3a}{2b} + \left(\frac{3a}{2b} + \frac{4}{\pi}\right) R(b) -$

$\frac{8}{\pi} R(-b),$

$\Gamma(k,y) = \int_y^\infty s^{k-1}e^{-s}ds$ is the upper incomplete gamma function and $\gamma_*(k,y) = \int_0^y s^{k-1}e^{-s}ds$ is the lower incomplete gamma function.

Proof: Since $T \sim GE(\mu,\sigma,k)$, then

$$\int_b^\infty tr(t)dt = \frac{1}{2^{k+1}\sigma\Gamma(k+1)}\int_b^\infty t\, e^{-\frac{1}{2}\left|\frac{t-\mu}{\sigma}\right|^{\frac{1}{k}}}dt.$$

Here, there are two cases:

Case 1: If $b \geq \mu$:

Let $y = \frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^{\frac{1}{k}} \Longrightarrow \sigma 2^k y^k + \mu = t \Longrightarrow dt = k\sigma 2^k y^{k-1}dy$, then

$$\int_b^\infty tr(t)dt = \frac{1}{2^{k+1}\sigma\Gamma(k+1)}\int_b^\infty te^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^{\frac{1}{k}}}dt = \frac{1}{2\Gamma(k)}\int_{\frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}}^\infty (\sigma 2^k y^k + \mu)e^{-y}y^{k-1}dy.$$

$$= \frac{\sigma 2^{k-1}}{\Gamma(k)}\int_{\frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}}^\infty y^{2k-1}e^{-y}dy + \frac{\mu}{2\Gamma(k)}\int_{\frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}}^\infty y^{k-1}e^{-y}dy.$$

$$= \frac{\sigma 2^{k-1}}{\Gamma(k)}\Gamma\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \frac{\mu}{2\Gamma(k)}\Gamma\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right)$$

where $\Gamma(k,y) = \int_y^\infty s^{k-1}e^{-s}ds$ is the upper incomplete gamma function. Hence,

$$\int_b^\infty tr(t)dt = \frac{1}{2\Gamma(k)}\left[\sigma 2^k \Gamma\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \mu\Gamma\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right)\right] < \infty.$$

Case 2: If $b < \mu$:

$$\int_b^\infty tr(t)dt = \int_b^\mu tr(t)dt + \int_\mu^\infty tr(t)dt = \int_b^\mu tr(t)dt + \frac{\mu}{2}.$$

Let $y = \frac{1}{2}\left(\frac{\mu-t}{\sigma}\right)^{\frac{1}{k}} \Rightarrow \mu - \sigma 2^k y^k = t \Rightarrow dt = -k\sigma 2^k y^{k-1}dy$, then

$$\int_b^\mu tr(t)dt = \frac{1}{2^{k+1}\sigma\Gamma(k+1)}\int_b^\mu te^{-\frac{1}{2}\left(\frac{\mu-t}{\sigma}\right)^{\frac{1}{k}}}dt = \frac{1}{2\Gamma(k)}\int_0^{\frac{1}{2}\left(\frac{\mu-b}{\sigma}\right)^{\frac{1}{k}}} (\mu - \sigma 2^k y^k)y^{k-1}e^{-y}dy.$$

$$= \frac{\sigma 2^{k-1}}{\Gamma(k)}\int_0^{\frac{1}{2}\left(\frac{\mu-b}{\sigma}\right)^{\frac{1}{k}}} y^{2k-1}e^{-y}dy + \frac{\mu}{2\Gamma(k)}\int_0^{\frac{1}{2}\left(\frac{\mu-b}{\sigma}\right)^{\frac{1}{k}}} y^{k-1}e^{-y}dy.$$

$$= \frac{\sigma 2^{k-1}}{\Gamma(k)}\gamma\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \frac{\mu}{2\Gamma(k)}\gamma_*\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right).$$

where $\gamma_*(k,y) = \int_0^y s^{k-1}e^{-s}ds$ is the lower incomplete gamma function.

Hence,

$$\int_b^\infty tr(t)dt = \frac{1}{2\Gamma(k)}\left[\sigma 2^k\gamma_*\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \mu\gamma_*\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right)\right] + \frac{\mu}{2} < \infty.$$

Now, by substituting in (2.13), we obtain the result in (3.21), where $\gamma_*(k,y)$ and $\Gamma(k,y)$ are bounded functions Alzer (1997).

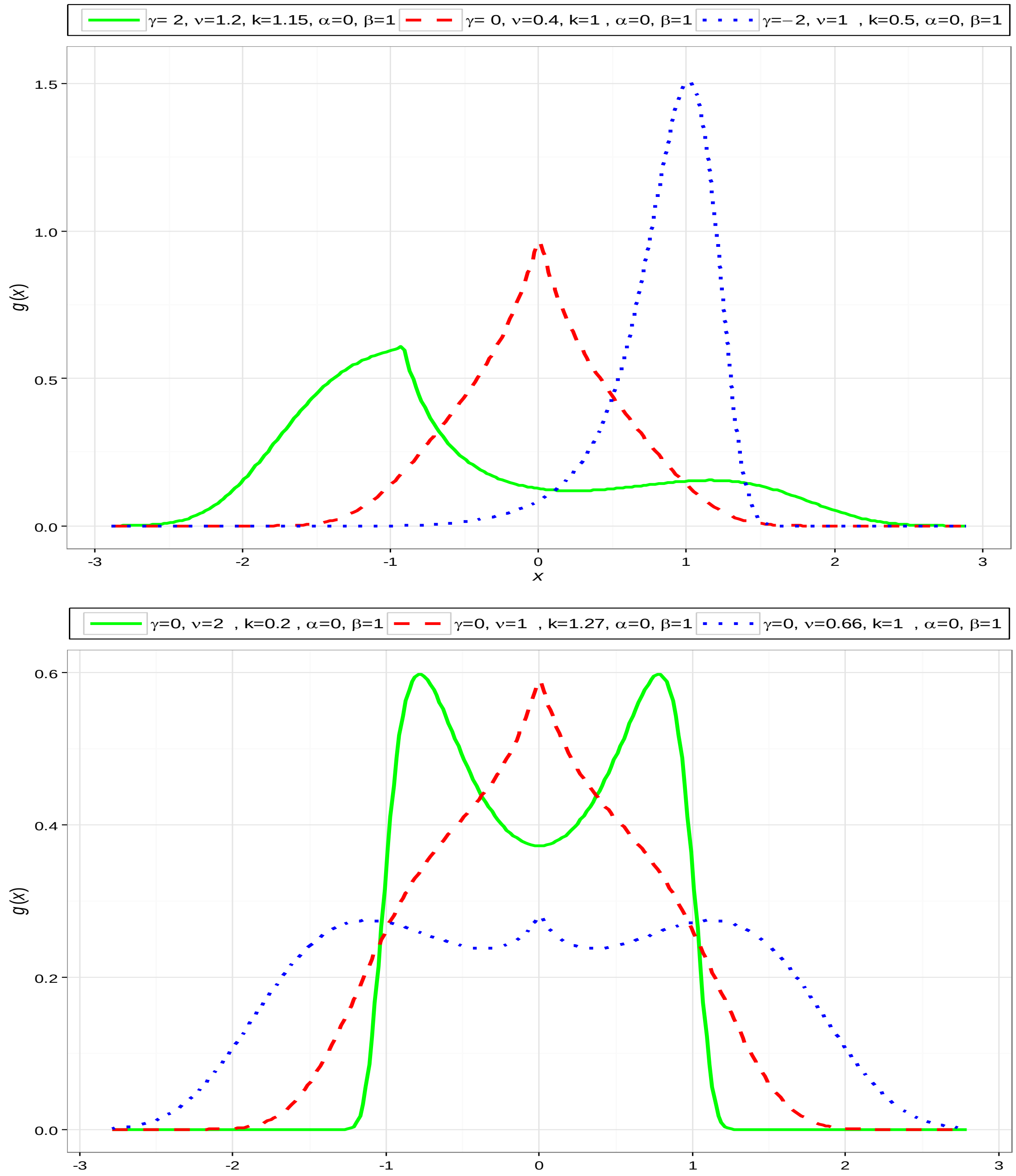
One example on this family, let the random variable X_f follow Generalized Hyperbolic Secant distribution with location parameter $-\infty < \alpha < \infty$ and scale parameter $\beta > 0$ as in **Sections 3.1** and **3.2**. From (3.17) we get

$$GEGHS(x;\boldsymbol{\theta}) = \frac{\pi}{2^{k+2}\beta\nu\Gamma(k+1)}\cosh\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right]e^{-\frac{1}{2}\left|\frac{1}{\nu}\left(\gamma+\sinh\left[\frac{\pi}{2}\left(\frac{x-\alpha}{\beta}\right)\right]\right)\right|^{\frac{1}{k}}} \tag{3.22}$$

where $x \in \mathbb{R}$, $\alpha, \gamma \in \mathbb{R}$ and $k, \beta, \nu > 0$, and $\boldsymbol{\theta} = (\gamma, \nu, k, \alpha, \beta)'$. The CDF of $GEGHS$is given by $G(x;\boldsymbol{\theta}) = G(x) = (1/2)\left[1 + \left(\mathrm{sgn}\left(\gamma + \sinh\left[(\pi/2)\left((x-\alpha)/\beta\right)\right]\right)/\Gamma(k)\right)\gamma\left(k, \left|\left[\gamma + \sinh\left[(\pi/2)\left((x-\alpha)/\beta\right)\right]\right]/\nu\right|^{1/k}\right)\right]$, The random variable X_g with PDF in (3.22) is said to be follow a five-parameter Generalized Error-Generalized Hyperbolic Secant distribution ($GEGHS$).

Plots in **Figure 3** show the $GEGHS$ density function for different parameter values, the distribution can be symmetric, right skewed, left skewed, unimodal, bimodal or trimodal.

Figure 3 The PDF of Generalized Error-Generalized Hyperbolic Secant distribution for various values of γ, ν, and k.



Lemma 15: Let the random variable X_g has $GEGHS$ distribution, then the non-central n^{th} moment $E\left[X_g^n\right]$ exists and satisfies the following inequality

$$E\left[\left|X_g\right|^n\right] < \begin{cases} \pi\left(\sum_{i=0}^{n}\binom{n}{i}\beta^i\alpha^{n-i}|E_i|\right) m_1(a,b,\mu,\sigma,k) & ;\text{if } b \geq \mu. \\ \pi\left(\sum_{i=0}^{n}\binom{n}{i}\beta^i\alpha^{n-i}|E_i|\right) m_2(a,b,\mu,\sigma,k) & ;\text{if } b < \mu, \end{cases} \tag{3.23}$$

where $E_i's$ are the Euler numbers,
$m_1(a,b,\mu,\sigma,k) =$
$= \frac{3}{4b\Gamma(k)}\left\{\sigma 2^k \Gamma\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \mu\Gamma\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right)\right\} - \frac{3a}{2b} + \left(\frac{3a}{2b} + \frac{4}{\pi}\right) R(b) - \frac{8}{\pi}\ R(-b),$
$m_2(a,b,\mu,\sigma,k) =$
$= \frac{3}{4b\Gamma(k)}\left\{\sigma 2^k \gamma\left(2k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \mu\gamma_*\left(k, \frac{1}{2}\left(\frac{b-\mu}{\sigma}\right)^{\frac{1}{k}}\right) + \frac{\mu}{2}\right\} - \frac{3a}{2b} + \left(\frac{3a}{2b} + \frac{4}{\pi}\right) R(b) -$
$\frac{8}{\pi}\ R(-b),$
$\Gamma(k,y) = \int_y^\infty s^{k-1}e^{-s}ds$ is the upper incomplete gamma function and $\gamma_*(k,y) = \int_0^y s^{k-1}e^{-s}ds$ is the lower incomplete gamma function.

Proof: By using the same steps used in proving **Lemma 6** above, and by substituting in (3.21), we obtain the result in (3.23).

Lemma 16: Let $0 < \lambda < 1$, the *quantile function* of the $GEGHS$ distribution defined in (3.22) is given by

$$Q_{GEGHS}(\lambda) = \alpha - \frac{2\beta}{\pi}\sinh^{-1}\left\{\gamma - \nu\ \mathrm{sign}(2\lambda - 1)\left(2Q_{gamma}(|2\lambda - 1|, k)\right)^k\right\} \qquad (3.24)$$

Proof: By equation (3.19) in **Lemma 12** above, since the quantile of the random variable X_f with CDF $F(x)$ is the quantile of the Generalized Hyperbolic Secant with parameters α and β, and it is given by $F^{-1}(\lambda;\alpha,\beta) = Q_{X_f}(\lambda;\alpha,\beta) = (\pi/2)\beta\sinh^{-1}\{\tan[\pi(\lambda - 1/2)]\} + \alpha$.

Since $\tan^{-1}(y) = \cot^{-1}(-y) - \pi/2$and $\sinh^{-1}(-y) = -\sinh^{-1}(y)$, the result in (3.24) follows.

4. Applications

We now consider two real numeric examples in order to demonstrate the usefulness of the $GGHS$ distribution defined in (3.14) and the $GEGHS$ distribution defined in (3.22) in fitting data sets.

4.1 The famous old faithful Geyser eruption data

The famous Old Faithful Geyser eruption data ($n = 272$) obtained from Härdle (1991, p. 201), this data is the duration time of eruption (in minutes) taken during August 1^{st} to August 15^{th}, 1985 (Dekking et al., 2005), and it is available in **faithful** data within ***MASS*** package in R 3.3.3 programming language (Venables and Ripley, 2002). **Figure 3** shows the Old Faithful Geyser eruption data histogram; it can be shown this data has two distinct modes (bimodal).

A common approach for fitting such a bimodal data is by using mixture distributions (Aljarrah et al., 2014). Arellano-Valle et al. (2010) used epsilon-skew-normal distribution to fit this data, they have gotten the same fitting results comparing with the mixture-normal distribution fitting results. The four-parameter $GGHS$ distribution defined in (3.1), the Mixture-Normal (MN) distribution, the five-parameter Normal-Weibull ($NW\{C\}$) distribution was defined by Aljarrah et al. (2014), and Beta-Normal (BN) distribution was defined by Eugene et al. (2002), are applied to fit the data using MLE procedure.

To compare the models, **Table 2** shows the MLE estimates and their standard errors, log-likelihood values, AIC (Akaike Information Criterion), $CAIC$ (Consistent Akaike Information Criterion), W (Durbin-Watson) test statistic, A (Anderson-Darling) test statistic, and K-S (Kolmogorov-Smirnov) test statistic with its corresponding p-value. In general, the smallest the values of: *log-likelihood*, AIC, $CAIC$, W, A, and K-S, and the largest the value of the K-S corresponding p-value, gives the best the fit to the data.

The results in **Table 2** indicate that the four-parameter $GGHS$ distribution outperforms the three distributions: MN, five-parameter $NW\{C\}$, and BN distributions, and gives the best fit based on the all six measures: *log-likelihood*, AIC, $CAIC$, W, A, and K-S statistic with its corresponding p-value.

Plots of the probability density functions: $GGHS$, MN, five-parameter $NW\{C\}$, and BN with ML estimate parameters versus the data, shown in **Figure 4.**
The $GGHS$ distribution can fit well wide variety of distribution shapes, including bimodal data such as Old Faithful Geyser eruption data.

Table 2: Parameter estimates (standard errors of the MLE in parentheses) and goodness-of-fit statistics for the famous Old Faithful Eruption data

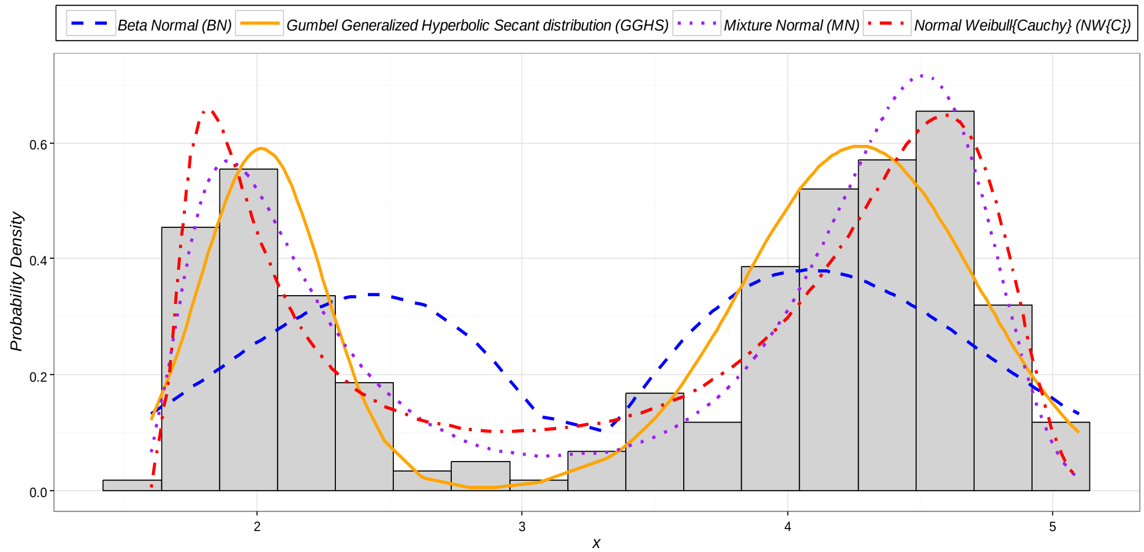
	Gumbel Generalized Hyperbolic Secant (GGHS)	**Five-parameter Normal Weibull {Cauchy} (NW{C})**	**Mixture-Normal ^{a} (MN)**	**Beta-Normal (BN)**
MLE (SE)	$\hat{\gamma} =$ 0.958 (1.409)	$\hat{\sigma} =$ 5.155 (0.645)	$\hat{\mu}_1 =$ 4.273 (0.034)	$\hat{\alpha} =$ 0.008 (0.002)
	$\hat{v} =$ 17.620 (3.706)	$\hat{c} =$ 2.032 (0.150)	$\hat{\sigma}_1 =$ 0.437 (0.027)	$\hat{\beta} =$ 0.007 (0.002)
	$\hat{\alpha} =$ 3.119 (0.022)	$\hat{\gamma} =$ 1.885 (0.116)	$\hat{\mu}_2 =$ 2.019 (0.026)	$\hat{\mu} =$ 3.219 (0.055)
	$\hat{\beta} =$ 0.569 (0.030)	$\hat{\mu} =$ 1.883 (0.446)	$\hat{\sigma}_2 =$ 0.236 (0.023)	$\hat{\sigma} =$ 0.071 (0.009)
	----	$\hat{\delta} =$ 1.342 (0.060)	$\hat{p} =$ 0.652 (0.029)	----
Log-likelihood	−265.971	−270.007	−276.360	−352.061
AIC	539.941	550.015	562.720	712.122
$CAIC$	540.091	550.240	562.946	712.271
W	0.079	0.110	0.175	7.681
A	0.564	0.680	1.349	45.418
K-S *statistic*	0.040	0.046	0.049	0.566
$p - value$	0.773	0.610	0.540	0

^{a}Mixture normal is defined as $p\,N(\mu_1, \sigma_1) + (1-p)\,N(\mu_2, \sigma_2)$.

4.2 Australian athletes' data

Cook and Weisberg (1994) proposed the Australian athletes' data, this data contains 13 variables on 102 male and 100 female Australian athletes collected at the Australian Institute of Sport. This data is available in **ais** data within ***DAAG*** package in R 3.3.3 programming language (John and W. John, 2015).

Figure 4 The PDFs for the famous Old Faithful Geyser eruption data.



There were some applications on this data, including:

1) Application on the generalized skew two-piece skew-normal distribution with the heights for the 100 female athletes and the hemoglobin concentration levels for the 202 athletes was applied by Jamalizadeh et al. (2011).

2) Application on the extended skew generalized normal distribution with 202 percentage of the hemoglobin blood cell for the male athletes was applied by Choudhury and Abdul Matin (2011).

3) Applications on the Gumbel-Weibull (GW) distribution with two variables in this data: the sum of skin folds (SSF) and the height in centimeters for the 100 female athletes were applied by Al-Aqtash et al. (2014). Additionally, the height in centimeters for the 100 Australian female athlete's data is unimodal and left skewed (skewness $= -0.560$, kurtosis $= 4.197$).

Al-Aqtash et al. (2014) have compared their new GW distribution MLE fits with the ML fits of the following distributions: the BN distribution, the Weibull-Pareto (WP) distribution defined by Alzaatreh et al. (2013), and the Exponentiated-Weibull (EW) distribution defined by Mudholkar and Srivastava (1993). They were found the GW distribution provides the best MLE fit comparing with the other compared distributions.

The five-parameter $GEGHS$ distribution defined in (3.14), the GW distribution, the EW distribution, and BN distribution are applied to fit the data using MLE procedure.

To compare the models, **Table 3** shows the MLE estimates and their standard errors, log-likelihood values, AIC, $CAIC$, W test statistic, A test statistic, and K-S test statistic with its corresponding p-value.

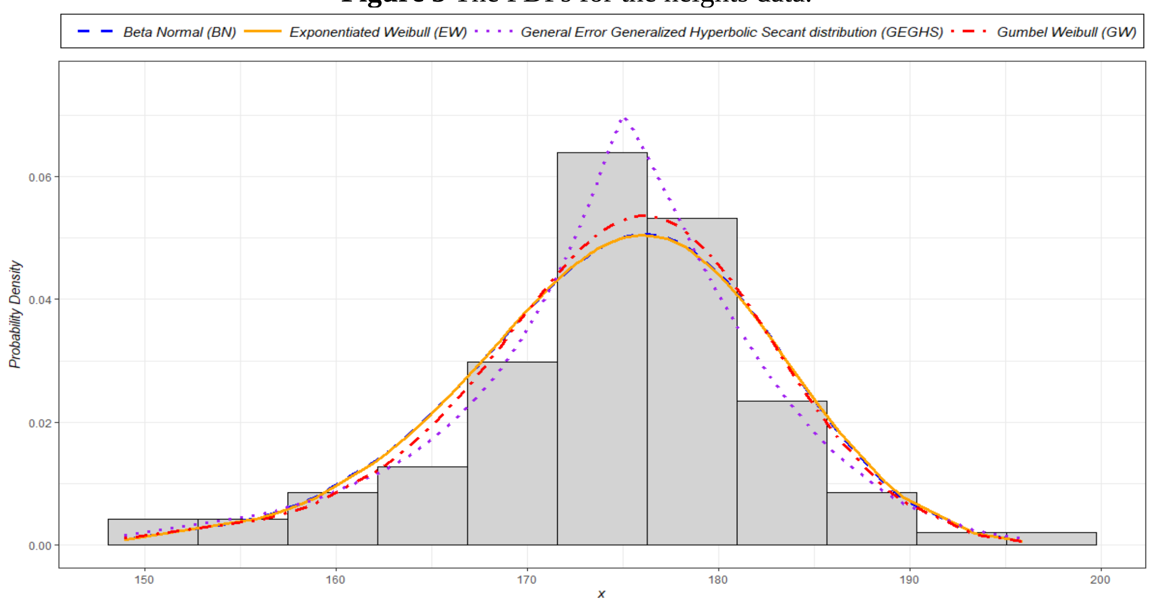
The results in **Table 3** indicate that the five-parameter $GEGHS$ distribution outperforms the three distributions: GW, EW, and BN distributions, and gives the best fit based on the all six measures: *log-likelihood*, AIC, $CAIC$, W, A, and K-S statistic with its corresponding p-value.

Plots of the probability density functions: $GEGHS$ distribution, GW distribution, EW distribution, and BN distribution with MLE estimate parameters versus the data, shown in **Figure 5**. The $GEGHS$ distribution can fit well wide variety of distribution shapes, including left skewed unimodal data such as the Australian athletes' data.

Table 3: Parameter estimates (standard errors of the MLE in parentheses) and goodness-of-fit statistics for the heights data

	General Error-Generalized Hyperbolic Secant (GEGHS)	Gumbel-Weibull (GW)	Exponentiated-Weibull (EW)	Beta-Normal (BN)
$MLE\ (SE)$	$\hat{\gamma} = -0.514\ (0.017)$	$\hat{a} = 12.268\ (2.153)$	$\hat{\alpha} = 14.741\ (3.236)$	$\hat{\alpha} = 0.982\ (1.371)$
	$\hat{\nu} = 0.222\ (0.076)$	$\hat{\lambda} = 147.018\ (12.582)$	$\hat{\theta} = 2.784\ (1.339)$	$\hat{\beta} = 8.391\ (25.138)$
	$\hat{k} = 0.879\ (0.160)$	$\hat{\mu} = 7.177\ (6.109)$	$\hat{\sigma} = 170.264\ (4.546)$	$\hat{\mu} = 194.059\ (29.049)$
	$\hat{\alpha} = 164.863\ (0.168)$	$\hat{\sigma} = 3.933\ (2.959)$	----	$\hat{\sigma} = 13.328\ (14.349)$
	$\hat{\beta} = 32.235\ (0.375)$	----	----	----
Log-likelihood	-347.971	-349.330	-350.369	-350.30
AIC	705.941	706.659	706.739	708.60
$CAIC$	706.580	707.081	706.989	709.02
W	0.030	0.060	0.094	0.093
A	0.185	0.383	0.592	0.584
K-S statistic	0.048	0.057	0.071	0.072
$p-value$	0.977	0.907	0.692	0.676

Figure 5 The PDFs for the heights data.



5. Summary and Conclusion

In this paper, we have proposed a method for generating a new family of univariate continuous distributions using the tangent function $W(F(x)) = a + b\tan\left(\pi(F(x) - 1/2)\right)$ of the CDF $F(x)$. In **Table 1** we have presented a list of some examples of the $T - X$ family of distributions based on the tangent function derived from different T distributions with support $(-\infty, \infty)$.

Two new distributions in the family: four-parameter Gumbel-Generalized hyperbolic secant distribution ($GGHS$) and five-parameter Generalized Error-Generalized hyperbolic secant distribution ($GEGHS$) are defined and some of their properties are given and discussed: quantiles, *Shannon entropy*, and existence of the n^{th} raw moment with its upper bound. The shapes of these distributions were found: skewed right, skewed left, or symmetric, and unimodal, bimodal, or trimodal.

To illustrate and assess the flexibility of the distributions, the $MLEs$ of the $GGHS$ distribution for the Old Faithful Geyser eruption data is computed, this data is bimodal data, and it was fitted by using: Mixture-Normal distribution, five-parameter $NW\{C\}$ distribution, and BN distribution (Aljarrah et al., 2014). Furthermore, the $GEGHS$ distribution has been used to fitted the Australian athletes' data, whereas this data set is unimodal and left skewed and it was fitted by using: GW distribution, EW distribution, and BN distribution (Al-Aqtash et al., 2014). The $GGHS$ and $GEGHS$ distributions have been found a very flexible and capable of fitting these data sets with the highest *Log-likelihood* value and the smallest AIC, $CAIC$, W, A, and K-S values among the four distributions.

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